KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLII

No. 10

President: J. VAN DER HOEVE

Secretary: M. W. WOERDEMAN

CONTENTS

Geophysics. — *On the vibrations of an elastic sphere with central core.*

By J. G. SCHOLTE. (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of November 25, 1939.)

§ 1. A homogeneous elastic body of unlimited extent can transmit two kinds of waves, the longitudinal and the transversal. When the body is limited, both kinds of waves are reflected by its boundary and give rise to other waves of the same kind. Any vibratory motion of the body can be represented as the result of superposing longitudinal and transversal waves. In case of an infinite solid with an infinite horizontal surface these waves can combine to form a displacement that does not penetrate far beneath the surface: the Rayleigh wave. Assuming that this body is covered by a layer with other elastic properties, it is possible to construct another kind of surface waves: the Love waves. Further the effect, due to gravity and the surface layer, on the Rayleigh waves — and various similar corrections — can be taken into account.

There is however another method of investigating the movements of a limited body: the method of normal functions. The solution of the equation of wave propagation on a sphere is then written as a combination of some spherical harmonics, satisfying the boundary conditions. The theory of the superficial waves must be included in this theory of the vibrations of a sphere, as has been pointed out by RAYLEIGH. In case of a homogeneous sphere the deduction of the equation, giving the velocity of the Rayleigh waves from the period equation for the vibrations of a sphere, has been effected by BROMWICH and LOVE.

In this paper we shall investigate the oscillations of a sphere with central core, taking gravity into account, and are to arrive at period equations, from which numerous known equations can be deduced.

§ 2. According to the theory of LOVE [3]), concerning the oscillations of a homogeneous sphere, the equations of vibratory motion are three of the type

$$\varrho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \triangle}{\partial x} + \mu \nabla^2 u + \varrho \frac{\partial}{\partial x}\left(A \frac{\partial V}{\partial r} \right) - \varrho \triangle \frac{\partial V}{\partial x} + \varrho \frac{\partial W}{\partial x}$$

where u = the x-component of the displacement; with similar equations for the y and z components (v and w). We have

ϱ = the density; λ and μ = Lamé's constants.

$A =$ the radial component of the displacement.

$\triangle = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.

$V =$ the potential if no disturbation.

$W =$ the potential due to a distribution of mass ($-\varrho\triangle$ in the medium and $+\varrho A$ at its boundary) together with that due to possible external bodies (e.g. the moon); hence

$\nabla^2 W = 4\pi\gamma\varrho\triangle$. ($\gamma =$ gravitation constant 6.7×10^{-8}).

These equations can be transformed into:

$$\left.\begin{aligned}
&\varrho\frac{\partial^2 (r\, rot_r)}{\partial t^2} = \mu\nabla^2 (r\, rot_r) \\
&\varrho\frac{\partial^2 \triangle}{\partial t^2} = (\lambda+2\mu)\nabla^2\triangle + 8\pi\gamma\varrho^2\triangle - \varrho\frac{\partial V}{\partial r}\cdot\frac{\partial\triangle}{\partial r} + \varrho\nabla^2\left(A\frac{\partial V}{\partial r}\right) \\
&\varrho\frac{\partial^2 (rA)}{\partial t^2} = \mu\nabla^2 (rA) + \varrho r\frac{\partial}{\partial r}\left(A\frac{\partial V}{\partial r}\right) + \\
&\qquad + (\lambda+\mu) r\frac{\partial\triangle}{\partial r} - 2\mu\triangle - \varrho\triangle r\frac{\partial V}{\partial r} + \varrho r\frac{\partial W}{\partial r}
\end{aligned}\right\} \quad . \quad (I)$$

where $r = \sqrt{x^2+y^2+z^2}$, $rot_r =$ the radial rotation component.

We shall write the solutions of this set of equations in the form $F(r) . W_n . e^{ipt}$; W_n is a spherical solid harmonic of degree n.

From the form of the equations (I) it will be seen, that there are two possible types of vibrations:

1^0. those, which involve no dilatation and no radial displacement; these correspond with the oscillations called by LAMB [1]) "vibrations of the first class".

2^0. the second type, which LAMB described as being of the "second class", are those which cause no radial rotation component.

§ 3. Beginning with the vibrations of the first class, we put $\triangle = 0$ and $A = 0$; equations (I) are then reduced to

$$(\varrho p^2 + \mu\nabla^2)(r\, rot_r) = 0.$$

The general solution is

$$r\, rot_r = n(n+1)(a_1\psi_n + b_1\pi_n) . W_n . e^{ipt},$$

where

$$\psi_n = \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}, \quad \pi_n = \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x},$$

and the argument $x = \frac{\varrho p^2}{\mu} r$ or αr.

As $\triangle = 0$ and $A = 0$:

$$\frac{\partial C}{\partial \varphi} + \frac{\partial (B \sin \psi)}{\partial \psi} = 0,$$

(B and C are the meridional and the azimuthal components of the displacement); hence

$$C = \frac{\partial M}{\partial \psi}, \quad B = -\frac{1}{\sin \psi} \frac{\partial M}{\partial \varphi},$$

which gives

$$r \, rot_r = r^2 \left(\nabla^2 - \frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} \right) . M$$

or

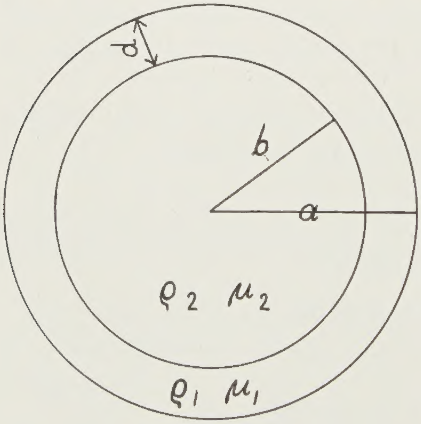
$$M = -(a_1 \psi_n + b_1 \pi_n) . W_n .$$

The components of the displacement are therefore:

$$A = 0, \; B = (a_1 \psi_n + b_1 \pi_n) . \frac{\partial W_n}{\sin \psi . \partial \varphi}, \; C = -(a_1 \psi_n + b_1 \pi_n) . \frac{\partial W_n}{\partial \psi}$$

(the time factor e^{ipt} being omitted for brevity).

We apply this solution to a sphere with central core; the movement of the first kind is perpendicular to the radius and must therefore be related to the Love waves, as those movements are also perpendicular to the vertical.



The solution is now:

$$\begin{cases} B = (a_1 \psi_n + b_1 \pi_n) . \dfrac{\partial W_n}{\sin \psi . \partial \varphi}, \\ C = -(a_1 \psi_n + b_1 \pi_n) . \dfrac{\partial W_n}{\partial \psi}, \\ \qquad \text{if } b \leqq r \leqq a; \text{ and} \end{cases}$$

$$\begin{cases} B = a_2 \psi_n' . \dfrac{\partial W_n}{\sin \psi . \partial \varphi}, \\ C = -a_2 \psi_n' . \dfrac{\partial W_n}{\partial \psi}, \\ \qquad \text{if } r \leqq b \end{cases}$$

(the argument of ψ_n' is $\frac{\varrho_2 p^2}{\mu_2} r$ or $\alpha_2 r$).

The boundary conditions are:

1. at $r = b$ continuity of movement:

$$a_1 \psi_n^b + b_1 \pi_n^b = a_2 \psi_n'^b$$

and continuity of tension:

$$\mu_1 \frac{\partial}{\partial b}\left\{(\mathrm{a}_1\, \psi_n^b + b_1\, \pi_n^b) \cdot \frac{W_n}{b}\right\} = \mu_2 \frac{\partial}{\partial b}\left\{\mathrm{a}_2\, \psi_n^{\prime b} \cdot \frac{W_n}{b}\right\}$$

2. the vanishing of the traction across the free surface $r = \mathrm{a}$:

$$\frac{\partial}{\partial \mathrm{a}}\left\{(\mathrm{a}_1\, \psi_n^a + b_1\, \pi_n^a) \cdot \frac{W_n}{\mathrm{a}}\right\} = 0.$$

Hence

$$b_1 = -\frac{(n-1)\, \psi_n^a + \alpha_1 \mathrm{a}\, \overline{\psi}_n^a}{(n-1)\, \pi_n^a + \alpha_1 \mathrm{a}\, \overline{\pi}_n^a} \cdot \mathrm{a}_1 \quad \text{and} \quad \mathrm{a}_2 = \frac{\psi_n^b + \frac{b_1}{\mathrm{a}_1} \cdot \pi_n^b}{\psi_n^{\prime b}} \cdot \mathrm{a}_1$$

(a horizontal line means differentiation to the argument).

Substituting in the second equation we obtain the period equation:

$$\frac{\mu_2}{\mu_1} \cdot \left\{(n-1) + \alpha_2\, b \cdot \frac{\overline{\psi}_n^{\prime b}}{\psi_n^{\prime b}}\right\} = (n-1) + \alpha_1\, b \cdot \frac{1 + \frac{b_1}{\mathrm{a}_1} \cdot \frac{\overline{\pi}_n^b}{\overline{\psi}_n^b}}{1 + \frac{b_1}{\mathrm{a}_1} \cdot \frac{\pi_n^b}{\psi_n^b}} \cdot \frac{\overline{\psi}_n^b}{\psi_n^b}$$

or:

$$\left\{\left(1 - \frac{\mu_1}{\mu_2}\right) \cdot (n-1) \cdot \psi_n^{\prime b} + \alpha_2\, b \cdot \overline{\psi}_n^{\prime b}\right\} \cdot$$
$$\cdot \left\{(n-1)\,(\psi_n^b\, \pi_n^a - \psi_n^a\, \pi_n^b) + \alpha_1 \mathrm{a}\, (\psi_n^b\, \overline{\pi}_n^a - \overline{\psi}_n^a\, \pi_n^b)\right\} =$$
$$= \frac{\varrho_1}{\varrho_2} \cdot \alpha_2\, b \cdot \psi_n^{\prime b} \cdot \left\{(n-1)\,(\overline{\psi}_n^b\, \pi_n^a - \psi_n^a\, \overline{\pi}_n^b) + \alpha_1 \mathrm{a}\, (\overline{\psi}_n^b\, \overline{\pi}_n^a - \overline{\psi}_n^a\, \overline{\pi}_n^b)\right\}.$$

By putting $a = b$ we arrive at an equation, identical with that found by LAMB [1]) for the vibrations of the 1[st] class of a homogeneous sphere:

$$\alpha_2\, \mathrm{a} \cdot \frac{\overline{\psi}_n^{\prime a}}{\psi_n^{\prime a}} + (n-1) = 0.$$

a. To explain seismometric data it is generally assumed, that the thickness ($d = a - b$) of the surface layer is small compared with the earth's radius. As a first approximation we have: $\psi_n^b = \psi_n^a - \alpha\, d\, \overline{\psi}_n^a$; then $\overline{\psi}_n^b = \overline{\psi}_n^a - \alpha\, d \cdot \overline{\overline{\psi}}_n^a$, and as ψ_n^a satisfies $\overline{\overline{\psi}}_n^a + \frac{2\,(n+1)}{\alpha\, a}\, \overline{\psi}_n^a + \psi_n^a = 0$:

$$\overline{\psi}_n^b = \overline{\psi}_n^a \cdot \left\{1 + 2\,(n+1)\, \frac{d}{a}\right\} + \alpha\, d \cdot \psi_n^a.$$

After some reduction we find the period equation in the form:

$$\alpha_2 a \cdot \frac{\overline{\psi}'^a_n}{\psi^a_n} + (n-1) = \frac{\left(1 - \frac{\mu_1}{\mu_2}\right)(n-1)(n+2) - \left(1 - \frac{\mu_2}{\mu_1} \cdot \frac{\varrho_1^2}{\varrho_2^2}\right) \cdot (\alpha_2 a)^2}{(2n+1)\frac{d}{a} + 1} \cdot \frac{d}{a} \quad (II)$$

if $n d \langle\langle a$.

In case of a sectorial harmonic $\frac{\pi a}{n}$ is the spherical distance between two points on the surface of the earth where $W_n = 0$, or $\frac{2\pi a}{n}$ is the "wave length". Therefore the condition $n d \langle\langle a$ means, that the wave length should be great compared with the depth of the layer; the vibrations of the earth are then those given by the period equation of a homogeneous sphere (consisting of the material of the earth's core), with a correction term due to the surface layer.

b. This approximation is no longer applicable, when the degree n of the spherical solid harmonic is great. Taking a, b and n very great, we shall find the period equation of the oscillations, possible in an infinite solid with a plane surface, covered by a surface layer with depth d, viz. the equation of the Love-waves. We suppose: a, b and n are infinite, so that $a - b = d$ and $\frac{n}{a} = l$ are finite. It is now necessary to obtain the form of ψ_n (or π_n), when both the order and the argument are very great and $r = b \pm z$, $z \langle\langle b$. Using the method of BROMWICH [2]):

ψ_n satisfies the equation:

$$\frac{d^2y}{dr^2} + \frac{2(n+1)}{r}\frac{dy}{dr} + \alpha_1^2 y = 0$$

write $r = b + z$:

$$\frac{d^2y}{dz^2} + \frac{2(n+1)}{z+b}\frac{dy}{dz} + \alpha_1^2 y = 0;$$

as $n \rangle\rangle 1$, $r \langle\langle b$:

$$\frac{d^2y}{dz^2} + 2l\frac{dy}{dz} + \alpha_1^2 y = 0; \text{ hence } y = e^{-lz \pm s_1 z}, \text{ where } s_1 = \sqrt{l^2 - \alpha_1^2}.$$

The functions must then be of the form:

$$\psi_n = e^{-lz}(c_1 e^{s_1 z} + d_1 e^{-s_1 z})$$
$$\pi_n = e^{-lz}(c_2 e^{s_1 z} + d_2 e^{-s_1 z}).$$

We have now (c_1, d_1, c_2, d_2 are unknown constants):

$$b_1 = -\frac{(n-1)\psi^a_n + \alpha_1 a \overline{\psi}^a_n}{(n-1)\pi^a_n + \alpha_1 a \overline{\pi}^a_n} \cdot a_1 \approx -\frac{l\psi^a_n + \left(\frac{d\overline{\psi}_n}{dz}\right)_{z=d}}{l\pi^a_n + \left(\frac{d\overline{\pi}_n}{dz}\right)_{z=d}} a_1 = -\frac{c_1 e^{s_1 d} - d_1 e^{-s_1 d}}{c_2 e^{s_1 d} - d_2 e^{-s_1 d}} \cdot a_1$$

and

$$\frac{n-1}{b}+\alpha_1 \cdot \frac{1+\frac{b_1}{a_1}\cdot\frac{\overline{\pi}_n^b}{\overline{\psi}_n^b}}{1+\frac{b_1}{a_1}\cdot\frac{\pi_n^b}{\psi_n^b}}\cdot\frac{\overline{\psi}_n^b}{\psi_n^b}=l+\frac{\left(\frac{d\psi_n}{dz}\right)_{z=0}+\frac{b_1}{a_1}\cdot\left(\frac{d\pi_n}{dz}\right)_{z=0}}{(\psi_n)_{z=0}+\frac{b_1}{a_1}(\pi_n)_{z=0}}\approx$$

$$\approx s_1 \cdot \frac{(c_1-d_1)+\frac{b_1}{a_1}(c_2-d_2)}{(c_1+d_1)+\frac{b_1}{a_1}(c_2+d_2)}=s_1\cdot\frac{e^{s_1 d}-e^{-s_1 d}}{e^{s_1 d}+e^{-s_1 d}}=-s_1 \cdot tgh\, s_1 d.$$

BROMWICH found on this method:

$$\psi_n' = c_3 \cdot e^{(l-s_2)z}, \text{ where } z=b-r \text{ and } s_2=\sqrt{l^2-\alpha_2^2};$$

the left-hand member of the period equation can now be reduced to $\frac{\mu_2 s_2}{\mu_1}$. Hence:

$$\frac{\mu_2 s_2}{\mu_1}=-s_1 \cdot tgh\, s_1 d.$$

This equation has only roots, if s_1 is imaginary; we assume therefore $\alpha_1 > l$ and put $s_1=\sqrt{\alpha_1^2-l^2}$. The period equation becomes

$$\frac{\mu_2 s_2}{\mu_1 s_1}=tg\, s_1 d\,.,$$

which is the equation of the Love-waves, which are only possible if

$$\alpha_1 > l > \alpha_2, \text{ or } \frac{\varrho_1}{\mu_1} > \left(\frac{l}{p}\right)^2 > \frac{\varrho_2}{\mu_2}.$$

§ 4. Proceeding to the movements of the second class we narrow down the problem to that of a core, surrounded by an incompressible liquid; this is to the seismologist one of the most interesting of these problems, viz. the interaction between the movements of the ocean and the earth's core.

Since the liquid is incompressible: $\triangle=0$, but as $\lambda=\infty$, $\lambda\triangle$ will be finite. Putting $\lambda\triangle=D$, we have the modified equations

$$\begin{cases} \nabla^2\left(D+\varrho_1 A\frac{\partial V}{\partial r}\right)=0 \quad , \quad \nabla^2 W=0 \\ \varrho_1 p^2 A+\frac{\partial}{\partial r}\left(D+\varrho_1 A\frac{\partial V}{\partial r}\right)+\varrho_1\frac{\partial W}{\partial r}=0. \end{cases}$$

Suppose $W=(a+b r^{-(2n+1)})\, W_n$ and $D+\varrho_1 A\frac{\partial V}{\partial r}=(c+d r^{-(2n+1)})\, W_n$, where $W_n=r^n \times$ spherical surface harmonic S_n.

Whe have then: $A = \dfrac{\partial F}{\partial r}$, and, with $rot_r = 0$ and $\triangle = 0$, we find

$$B = \frac{1}{r}\frac{\partial F}{\partial \psi},\; C = \frac{1}{r \sin\psi}\frac{\partial F}{\partial \varphi}, \text{ with } F = -\frac{a\varrho_1 + c}{\varrho_1 p^2} \,.\, r^n S_n - \frac{b\varrho_1 + d}{\varrho_1 p^2} r^{-(n+1)} \,.\, S_n.$$

Assuming for simplicity's sake, that the core is also incompressible, the equations of motion of the core are

$$\begin{cases} \nabla^2 \left(D + \varrho_2 A \dfrac{\partial V}{\partial r} \right) = 0 \quad , \quad \nabla^2 W = 0 \\ (\varrho_2 p^2 + \mu \nabla^2)\, r A + r \dfrac{\partial}{\partial r} \left(D + \varrho_2 A \dfrac{\partial V}{\partial r} \right) + \varrho_2 r \dfrac{\partial W}{\partial r} = 0. \end{cases}$$

Put $W = a' W_n$ and $D + \varrho_2 A \dfrac{\partial V}{\partial r} = c' W_n$; then we have

$$(\varrho_2 p^2 + \mu \nabla^2)\, r A + r \frac{\partial}{\partial r} (a' \varrho_2 + c')\, W_n = 0,$$

whence $A = -\dfrac{a'\varrho_2 + c'}{\varrho_2 p^2} n W_n + n e \psi_n W_n$, where the argument of ψ_n is $\dfrac{\varrho_2 p^2}{\mu} r$ or $k^2 r$. Again we have $rot_r = 0$ and $\triangle = 0$, therefore:

$$B = \left\{ -\frac{a'\varrho_2 + c'}{\varrho_2 p^2} + e \,.\, \left(\frac{1}{n+1} kr \overline{\psi}_n + \psi_n \right) \right\} . \frac{1}{r}\frac{\partial W_n}{\partial \psi}$$

$$C = \text{the same factor} \times \frac{1}{r \sin\psi}\frac{\partial W_n}{\partial \varphi}.$$

The boundary conditions at the bottom of the ocean ($r = r_2$) are the continuity of W, A and of the tensions T_{RR} and $T_{R\psi}$, or, in the same order:

1. $a' = a + b r_2^{-(2n+1)}$.

2. $-n.\dfrac{a'\varrho_2 + c'}{\varrho_2 p^2} + n e \psi_n = -n \,.\, \dfrac{a\varrho_1 + c}{\varrho_1 p^2} + (n+1) \,.\, \dfrac{b\varrho_1 + d}{\varrho_1 p^2} r_2^{-(2n+1)}$.

3. $c' + \left({}^4/_3 \pi \gamma \varrho_2^2 + 2\mu \dfrac{n-1}{r_2^2} \right) . r_2 A + 2\mu \,.\, \dfrac{n}{r_2^2} \,.\, e k r_2 \,.\, \overline{\psi}_n =$

$$= c + d r_2^{-(2n+1)} + {}^4/_3 \pi \gamma \varrho_1 \varrho_2 r_2 A.$$

4. $2(n-1) \,.\, \left(-\dfrac{a'\varrho_2 + c'}{\varrho_2 p^2} + e \psi_n \right) - \dfrac{k^2 r_2^2}{n+1} e \psi_n - \dfrac{2}{n+1} e k r_2 \overline{\psi}_n = 0.$

The equations 2. and 4. give

$$e = \frac{2(n^2-1)}{n k r_2 (k r_2 \psi_n + 2\overline{\psi}_n)} \cdot r_2 A,$$

substituting in 4.:

$$\frac{c'}{\varrho_2 p^2} = -\frac{a'}{p^2} + \left\{ \frac{2(n^2-1)\psi_n}{n k r_2 (k r_2 \psi_n + 2\overline{\psi}_n)} - \frac{1}{n} \right\} \cdot r_2 A$$

with 3.

$$c + d\, r_2^{-(2n+1)} = \left\{ {}^4/_3\, \pi \gamma \varrho_2 (\varrho_2 - \varrho_1) + 2\mu \cdot \frac{n-1}{r_2^2} + \right.$$

$$\left. + \frac{2(n^2-1)}{k^2 r_2^2} \cdot \varrho_2 p^2 \cdot \frac{x + \frac{1}{n}}{x+1} - \frac{\varrho_2 p^2}{n} \right\} \cdot r_2 A$$

where we have put $2 \dfrac{\overline{\psi}_n}{k r_2 \psi_n} = x$ for brevity.

With 1. we find

$$\frac{\varrho_1}{\varrho_2} \cdot \frac{c + d\, r_2^{-(2n+1)} + \varrho_2 a + \varrho_2 b\, r_2^{-(2n+1)}}{\varrho_1 p^2 \cdot r_2 A} = \frac{{}^4/_3 \pi \gamma (\varrho_2 - \varrho_1)}{p^2} - \frac{1}{n} +$$

$$+ \frac{2(n-1)}{k^2 r_2^2} + \frac{2(n^2-1)}{k^2 r_2^2} \cdot \frac{x + \frac{1}{n}}{x+1}$$

or

$$2 \frac{\overline{\psi}_n}{k r_2 \psi_n} = - \frac{\dfrac{k^2 r_2^2}{2(n-1)} H - (2n+1)}{\dfrac{k^2 r_2^2}{2(n-1)} H - n(n+2)}, \quad \ldots \ldots \quad (IIIa)$$

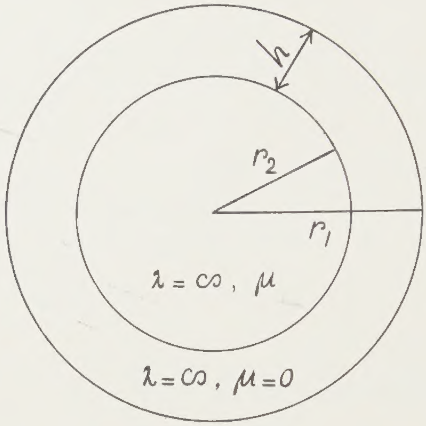
where

$$H = 1 - \frac{{}^4/_3 \pi \gamma (\varrho_2 - \varrho_1)}{p^2} + n \cdot \frac{\varrho_1}{\varrho_2} \cdot \frac{c + d\, r_2^{-(2n+1)} + \varrho_2 a + \varrho_2 b\, r_2^{-(2n+1)}}{-(c + \varrho_1 a) n + (d + \varrho_1 b)(n+1) r_2^{-(2n+1)}}.$$

The boundary condition at the free surface ($r = r_1$) is $T_{RR} = 0$; hence:

$c + d\, r_1^{-(2n+1)} + {}^4/_3\, \pi \gamma \varrho_1 \varrho_0 r_1 A = 0$, where we have put $\varrho_0 = \varrho_1 + (\varrho_2 - \varrho_1) \cdot \left(\dfrac{r_2}{r_1}\right)^3$.

Again there are two boundary conditions due to the surface mass-



distribution $\varrho_1 A$ at $r=r_1$ and $(\varrho_1-\varrho_2) A$ at $r=r_2$. The potential W is

$$(a+b\, r^{-(2n+1)})\,.\, r^n S_n, \text{ if } r_2 \leqq r \leqq r_1$$

$$a'\, r^n S_n \text{ or } (a+b\, r_2^{-(2n+1)})\,.\, r^n S_n, \text{ if } r \leqq r_2$$

and at external points $(r \geqq r_1)$

$$(a\, r_1^{+(2n+1)}+b)\,.\, r^{-(n+1)}\,.\, S_n.$$

The surface characteristic equations for the potential are therefore:

$$\frac{\partial}{\partial r}\{(a+b\, r^{-(2n+1)})\, W_n\{ - \frac{\partial}{\partial r}\{(a\, r_1^{2n+1}+b)\, r^{-(2n+1)}.\, W_n\} = 4\pi\gamma\varrho_1 A \text{ at } r=r_1$$

$$\frac{\partial}{\partial r}\{(a+b\, r^{-(2n+1)})\, W_n\} - \frac{\partial}{\partial r}\{(a+b\, r_2^{-(2n+1)})\, W_n\} = 4\pi\gamma(\varrho_1-\varrho_2)A \text{ at } r=r_2.$$

Or

$$(2n+1).\, b\, r_2^{-(2n+1)} = 4\pi\gamma(\varrho_2-\varrho_1).\, r_2 A \text{ and } (2n+1)a = 4\pi\gamma\varrho_1 r_1 A.$$

These equations give with 6.

$$b = a.\frac{\varrho_2-\varrho_1}{\varrho_1}.\frac{1+\dfrac{n(n+1)(1-s)}{2n+1}.\dfrac{f^\alpha}{\alpha s}}{1+\dfrac{n(n+1)(1-s)}{2n+1}.\dfrac{1}{\beta}}.\, r_2^{2n+1},$$

where

$$\alpha = \frac{(2n+1)\, p^2}{4\pi\gamma\varrho_1},\ \beta = \frac{(2n+1)\, p^2}{4\pi\gamma\,(\varrho_2-\varrho_1)},\ s = \left(\frac{r_2}{r_1}\right)^{2n+1},$$

$$f^\alpha = 1 - \frac{2n+1}{3}\frac{\varrho_0}{\varrho_1} + \frac{\alpha}{n},\ f^\beta = \frac{\varrho_2-\varrho_1}{\varrho_1} - \frac{2n+1}{3}\frac{\varrho_2}{\varrho_1} + \frac{\alpha}{n}.$$

Substituting in H we obtain:

$$H = n\frac{\varrho_1}{\varrho_2}\cdot\frac{\dfrac{f^\beta}{\beta}+\dfrac{f^\alpha}{\alpha s}+\dfrac{2}{\beta}-\dfrac{n(n+1)(1-s)}{2n+1}\cdot\dfrac{1}{\beta^2}}{1+\dfrac{n(n+1)(1-s)}{2n+1}\cdot\dfrac{f^\alpha}{\alpha s}} \quad . \quad . \quad . \quad (III\,b)$$

Equations $(III\,a)$ and $(III\,b)$ give the period equation of a sphere, covered by an ocean of uniform depth.

a. It is obvious, that we can consider the periods p, determined by a known value of n, as the periods of the vibrations of the core, altered by the surface layer, or as those of the ocean with a variation due to the core. Equation (III) must therefore include as limiting cases the period equations, both of a free oscillating homogeneous sphere and of a free oscillating ocean.

Firstly: we put $r_1 = r_2$ or $s = 1$; then

$$H = n\frac{\varrho_1}{\varrho_2}\left(\frac{f^\beta}{\beta}+\frac{f^\alpha}{\alpha}+\frac{2}{\beta}\right)$$

$$H = 1 - \frac{2n(n-1)}{(2n+1)p^2}\cdot\tfrac{4}{3}\pi\gamma\varrho_2.$$

Substituting in $(IIIa)$ we obtain

$$2\frac{\overline{\psi}_n}{k r_2 \psi_n} = -\frac{\dfrac{k^2 r_2^{\,2}}{2(n-1)}-\dfrac{n g \varrho_2 r_2}{(2n+1)\mu}-(2n+1)}{\dfrac{k^2 r_2^{\,2}}{2(n-1)}-\dfrac{n g \varrho_2 r_2}{(2n+1)\mu}-n(n+2)}, \text{ where } g = {}^4/_3\,\pi\gamma\varrho_2 r_2,$$

the equation found by BROMWICH [2]) in case of a homogeneous incompressible sphere.

Secondly: to deduce the period equation of an ocean with uniform depth, covering a non-vibrating core, we must assume that $\mu = \infty$. Then $k = 0$ and $H = \infty$, or

$$1 + \frac{n(n+1)(1-s)}{2n+1}\cdot\frac{f^\alpha}{\alpha s} = 0;$$

this gives:

$$p^2 = \frac{n(n+1)(1-s)}{n+1+ns}\cdot\left(1-\frac{3}{2n+1}\cdot\frac{\varrho_1}{\varrho_0}\right)\cdot\tfrac{4}{3}\pi\gamma\varrho_0,$$

being the equation in question. (LAMB. Hydrodynamics).

Note. Taking this solution of LAMB we find that the tension, due to the movement of the ocean, on the surface of the core

$$T_{RR} = \left(\frac{r_2}{r_1}\right)^n\cdot g\cdot\frac{2n+1}{n+1+s}\cdot\left(\varrho_1\cdot\frac{3n(1-s)}{(2n+1)^2}-\varrho_0\right)\cdot S_n.$$

This periodic tension causes a vibration of the core, which can only be neglected, if the wave length is small compared with the depth of the ocean.

b. In case of a liquid core we put $\mu = 0$; then $k = \infty$ and $H = 0$, or

$$\alpha^2\left(1 + \frac{\varrho_2 - \varrho_1}{\varrho_1} \cdot \frac{n+1+ns}{2n+1}\right) + n\alpha \Big\{ 1 - \frac{2n+1}{3} \frac{\varrho_0}{\varrho_1} +$$

$$+ \frac{\varrho_2 - \varrho_1}{\varrho_1} s - (n+1)(1-s) \frac{\varrho_2 - \varrho_1}{\varrho_1} \cdot \frac{1}{3} \frac{\varrho_0}{\varrho_1} - 2(n-1) \cdot \frac{\varrho_2 - \varrho_1}{\varrho_1} \cdot \frac{n+1+ns}{2n+1} \cdot \frac{1}{3} \frac{\varrho_2}{\varrho_1} \Big\}$$

$$+ \frac{n^2(n+1)(1-s)}{2n+1} \cdot \frac{\varrho_2 - \varrho_1}{\varrho_1} \cdot \Big\{ \left(1 - \frac{2n+1}{3} \frac{\varrho_0}{\varrho_1}\right) \left(\frac{\varrho_2 - \varrho_1}{\varrho_1} - \frac{2n+1}{3} \frac{\varrho_2}{\varrho_1}\right) - \frac{\varrho_2 - \varrho_1}{\varrho_1} s \Big\} = 0.$$

If n is great, the two roots of this equation correspond with 1. the frequency of an ocean of uniform depth, and 2. the frequency of two infinite liquids with a plane interface, as has been remarked by SEZAWA [8]).

c. BROMWICH (and later LOVE) has proved, that the Rayleigh-waves are vibrations of the 2nd class of a homogeneous sphere (if r_1 and n are very great). We can therefore expect, that the period equation of the corresponding vibrations of a heterogeneous sphere include as a limiting case the equation which determines the propagation of the Rayleigh-waves, with correction terms due to gravity and the liquid surface layer.

When r_1, r_2 and n are infinite, so that $r_1 - r_2 = h$ and $\frac{n}{r_1} = l$ are finite, then:

$$s = \left(\frac{r_2}{r_1}\right)^{2n+1} \approx e^{-2hl} \quad , \quad \varrho_0 = \varrho_1 + (\varrho_2 - \varrho_1)\left(\frac{r_2}{r_1}\right)^3 \approx \varrho_2.$$

$$\alpha = \frac{(2n+1)\,p^2}{4\pi\gamma\varrho_1} \approx \frac{{}^2/_3\, n^2 p^2}{g\,l} \cdot \frac{\varrho_2}{\varrho_1}, \text{ where } g = {}^4/_3\,\pi\gamma\varrho_2 r_1.$$

After some reduction, we find

$$\frac{f^\alpha}{\alpha} \approx \frac{1}{n}\left(1 - \frac{g\,l}{p^2}\right) \quad \text{and} \quad \frac{f^\beta}{\beta} \approx \frac{\varrho_2 - \varrho_1}{\varrho_1} \cdot \frac{1}{n}\left(1 - \frac{g\,l}{p^2}\right).$$

Hence

$$H \approx \left(1 + \frac{\varrho_1}{\varrho_2} \cdot \frac{1 + \frac{gl}{p^2}}{\operatorname{ctgh} h\,l - \frac{gl}{p^2}}\right)\left(1 - \frac{gl}{p^2}\right), \text{ or } H = (1+K)\left(1 - \frac{g\,l}{p^2}\right).$$

Applying the method of BROMWICH to equation (*IIIa*) we find

$$4\sqrt{1-\zeta} - (2-\zeta)^2 + \zeta^2(1-H) = 0, \text{ where } \zeta = \frac{k^2}{l^2}.$$

Substituting H, the period equation becomes

$$4\sqrt{1-\zeta}-(2-\zeta)^2+\frac{g\,\varrho_2}{\mu\, l}\zeta(1+K)-\zeta^2 K=0,\quad K=\frac{\varrho_1}{\varrho_2}\cdot\frac{1+\dfrac{gl}{p^2}}{ctgh\, hl-\dfrac{gl}{p^2}}\,.\qquad (IV)$$

This equation determines the velocity of the Rayleigh-waves on the bottom of the ocean; it is obvious, that it includes the equation of the Rayleigh-waves ($g=0$ and $h=0$) and the equation

$$4\sqrt{1-\zeta}-(2-\zeta)^2+\frac{g\,\varrho_2}{\mu\, l}\zeta-\zeta^2\, tgh\, hl=0,$$

found by BROMWICH in case $hl \ll 1$.

I want to express my gratitude to Prof. VAN DER WAALS for his kind assistance and interest he has taken in this paper.

LITERATURE.

1. LAMB, Proc. Math. Soc. London, **13**, 1881.
(discussing the two possible kinds of oscillation of a homogeneous sphere, without taking self-gravitation into account).
2. BROMWICH, Proc. Math. Soc. London, **30**, 1899.
(dealing with an incompressible, homogeneous sphere, taking self-gravitation into account).
3. LOVE, "Some problems of Geodynamics", 1911.
(a compressible sphere, with self-gravitation).
4. JEANS, Proc. Roy. Soc. **102**, 1923.
(dealing with the same problem as LOVE, and considering mainly the limits of the solutions found).
5. MATUZAWA, Proc. Math. Soc. Tokyo, 1928.
6. SEZAWA, Bull. Earthq. Res. Inst. Tokyo, 1937.
(a heterogeneous sphere with central core; with simplifying suppositions as regards to the self-gravitation effect).
7. LAMB, Hydrodynamics, (Surface Waves).
(ocean with immovable core).
8. SEZAWA, Bull. Earthq. Res. Inst. Tokyo, 1938.
(ocean with liquid core).

Mathematics. — *Ueber Produkte von* LEGENDRE*schen Funktionen.* Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of November 25, 1939.)

§ 1. Bei der Definition der zugeordneten LEGENDREschen Funktion erster Art unterscheidet man gewöhnlich zwei Fälle [1]):

1. Es sei $w \neq \pm 1$, $|\arg(w+1)| < \pi$ und $|\arg(w-1)| < \pi$; die zugeordnete LEGENDREsche Funktion erster Art $P_n^m(w)$ wird definiert durch

$$P_n^m(w) = \frac{(w+1)^{\frac{1}{2}m}(w-1)^{-\frac{1}{2}m}}{\Gamma(1-m)}\, {}_2F_1(-n, 1+n; 1-m; \tfrac{1}{2}-\tfrac{1}{2}w).$$

2. Es sei $-1 < w < 1$; die zugeordnete LEGENDREsche Funktion erster Art $\mathbf{P}_n^m(w)$ wird definiert durch [2])

$$\mathbf{P}_n^m(w) = \frac{(1+w)^{\frac{1}{2}m}(1-w)^{-\frac{1}{2}m}}{\Gamma(1-m)}\, {}_2F_1(-n, 1+n; 1-m; \tfrac{1}{2}-\tfrac{1}{2}w). \quad . \quad (1)$$

Die zugeordnete LEGENDREsche Funktion zweiter Art $Q_n^m(w)$ kann erklärt werden durch [3])

$$Q_n^m(w) = \frac{2^{-n-1}\, e^{m\pi i}\sqrt{\pi}\,\Gamma(1+m+n)}{\Gamma(\frac{3}{2}+n)}(w+1)^{-\frac{1}{2}m}(w-1)^{\frac{1}{2}m-n-1}$$
$$\times\, {}_2F_1\left(1+n, 1-m+n; 2+2n; \frac{2}{1-w}\right);$$

hierin wird $w \neq \pm 1$, $|\arg(w+1)| < \pi$ und $|\arg(w-1)| < \pi$ vorausgesetzt [4]).

In dieser Arbeit setze ich stets

$$U_n^m(w) = \frac{e^{-m\pi i}}{\Gamma(1+m+n)}\, Q_n^m(w);$$

---

[1]) Man vergl. HOBSON, [**1**], 188 und 227.

[2]) Die Funktion $P_n^m(w)$ hat einen Sinn in der von $w=1$ bis $w=-\infty$ aufgeschnittenen w-Ebene; die Funktion $\mathbf{P}_n^m(w)$ dagegen wird nur für $-1 < w < 1$ definiert.

[3]) Man vergl. HOBSON, [**1**], 203, Formel (28).

[4]) Die Funktion zweiter Art brauche ich hier nur für $|\arg(w \pm 1)| < \pi$; eine (1) entsprechende Funktion zweiter Art kommt in dieser Note nicht vor.

zur Abkürzung setze ich ferner noch

$$C_n^m(w) = \tfrac{1}{2}\{(m^2-n^2)\, U_n^m(w)\, U_{-n}^m(w) + U_{n-1}^m(w)\, U_{-n-1}^m(w)\}$$

und

$$D_n^m(w) = \tfrac{1}{2}\{(m+n)\, U_n^m(w)\, U_{-n-1}^{m-1}(w) - (1-m+n)\, U_n^{m-1}(w)\, U_{-n-1}^m(w)\}.$$

§ 2. Ist $|\arg w| < \pi$ und genügen α, β, σ und τ den Bedingungen

$$\Re(\sigma) > 0,\ \Re(\tau) > 0 \text{ und } \Re(\alpha+\beta-\sigma-\tau) > 0,$$

so hat man, wie ich neuerdings [5]) bewiesen habe,

$$\left.\begin{aligned} \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\, {}_3F_2\begin{pmatrix} \lambda, \sigma, \tau; \\ \alpha, \beta; -w \end{pmatrix} &= \frac{1}{\Gamma(\sigma)\,\Gamma(\tau)\,\Gamma(\alpha+\beta-\sigma-\tau)} \int_0^1 (1+wu)^{-\lambda} \\ &\times {}_2F_1(\alpha-\tau, \beta-\tau; \alpha+\beta-\sigma-\tau; 1-u)\,(1-u)^{\alpha+\beta-\sigma-\tau-1}\, u^{\sigma-1}\, du. \end{aligned}\right\} \quad (2)$$

Mit Hilfe dieser Beziehung kann man Integraldarstellungen ableiten für gewisse Produkte von LEGENDREschen Funktionen.

Ist $z \neq 0$ und $|\arg z| < \frac{1}{2}\pi$, so gilt nämlich [6])

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{2^{2m} z^{-2m}}{\Gamma^2(1-m)}\, {}_3F_2\begin{pmatrix} \frac{1}{2}-m, -m-n, 1-m+n; \\ 1-m, 1-2m; -z^2 \end{pmatrix}, \quad . \quad . \quad (3)$$

$$P_n^m(\sqrt{1+z^2})\, P_n^{m-1}(\sqrt{1+z^2}) = \frac{2^{2m-1} z^{1-2m} \sqrt{1+z^2}}{\Gamma(1-m)\,\Gamma(2-m)}\, {}_3F_2\begin{pmatrix} \frac{3}{2}-m, 1-m-n, 2-m+n; \\ 2-m, 2-2m; -z^2 \end{pmatrix}, \quad (4)$$

$$P_n^m(\sqrt{1+z^2})\, P_{n-1}^m(\sqrt{1+z^2}) = \frac{2^{2m} z^{-2m} \sqrt{1+z^2}}{\Gamma^2(1-m)}\, {}_3F_2\begin{pmatrix} \frac{1}{2}-m, 1-m-n, 1-m+n; \\ 1-m, 1-2m; -z^2 \end{pmatrix}, \quad . \quad (5)$$

$$U_n^m(\sqrt{1+z^2})\, U_{n-1}^m(\sqrt{1+z^2}) = \frac{\pi}{2z\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{3}{2}+n)}\, {}_3F_2\begin{pmatrix} \frac{1}{2}-m, \frac{1}{2}+m, \frac{1}{2}; \\ \frac{1}{2}-n, \frac{3}{2}+n; -z^{-2} \end{pmatrix}, \quad . \quad . \quad (6)$$

$$C_n^m(\sqrt{1+z^2}) = \frac{\pi\sqrt{1+z^{-2}}}{2\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{1}{2}+n)}\, {}_3F_2\begin{pmatrix} \frac{1}{2}-m, \frac{1}{2}+m, \frac{1}{2}; \\ \frac{1}{2}-n, \frac{1}{2}+n; -z^{-2} \end{pmatrix}, \quad (7)$$

$$D_n^m(\sqrt{1+z^2}) = \frac{\pi(m-\frac{1}{2})\sqrt{1+z^{-2}}}{2z\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{3}{2}+n)}\, {}_3F_2\begin{pmatrix} \frac{3}{2}-m, \frac{1}{2}+m, \frac{1}{2}; \\ \frac{1}{2}-n, \frac{3}{2}+n; -z^{-2} \end{pmatrix}. \quad (8)$$

Durch Anwendung von (2) auf die rechten Seiten dieser Relationen

5) Man vergl. [5], Formel (16).

6) [2], 487; [4], 400; [3], 216. Für (6), (7) und (8) vergl. man [3], Formeln (113) und (57), Formeln (114) und (63) und Formeln (115) und (60).

bekommt man die verlangten Integraldarstellungen. Aus (3) und (2) folgt z.B., falls

$$z \neq 0,\ |\arg z| < \tfrac{1}{2}\pi,\ \Re(m+n) < 0 \text{ und } \Re(m-n) < 1 \quad . \quad . \quad (9)$$

ist [7])

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{z^{-2m}\,\Gamma(\frac{1}{2}-m)}{\sqrt{\pi}\,\Gamma(-m-n)\,\Gamma(1-m+n)\,\Gamma(1-m)}\int_0^1 (1+z^2u)^{m-\frac{1}{2}}$$

$$\times\, {}_2F_1(-n, -m-n; 1-m; 1-u)\,(1-u)^{-m}\,u^{-m-n-1}\,du;$$

setzt man hierin $u = \cosh^{-2} t$, so erhält man

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{2z^{-2m}\,\Gamma(\frac{1}{2}-m)}{\sqrt{\pi}\,\Gamma(-m-n)\,\Gamma(1-m+n)\,\Gamma(1-m)}\int_0^\infty (z^2+\cosh^2 t)^{m-\frac{1}{2}}$$

$$\times\, {}_2F_1(-n, -m-n; 1-m; \operatorname{tgh}^2 t)\,\sinh^{1-2m} t\,\cosh^{2m+2n} t\,dt.$$

Es gilt aber [8])

$${}_2F_1(-n, -m-n; 1-m; \operatorname{tgh}^2 t) = \Gamma(1-m)\,\sinh^m t\,\cosh^{-m-2n} t\,P_n^m(\cosh 2t).$$

Sind die Bedingungen (9) erfüllt, so besitzt daher die Funktion $\{P_n^m(\sqrt{1+z^2})\}^2$ die Integraldarstellung

$$\left.\begin{aligned}\{P_n^m(\sqrt{1+z^2})\}^2 &= \frac{2z^{-2m}\,\Gamma(\frac{1}{2}-m)}{\sqrt{\pi}\,\Gamma(-m-n)\,\Gamma(1-m+n)}\\ &\times\int_0^\infty (z^2+\cosh^2 t)^{m-\frac{1}{2}}\,P_n^m(\cosh 2t)\,\sinh^{1-m} t\,\cosh^m t\,dt.\end{aligned}\right\} \quad . \quad (10)$$

Eine mit (10) verwandte Integraldarstellung für das Produkt $U_n^m(\sqrt{1+z^2})\,U_{-n-1}^m(\sqrt{1+z^2})$ ist

$$\left.\begin{aligned}U_n^m(\sqrt{1+z^2})\,U_{-n-1}^m(\sqrt{1+z^2}) &= \frac{\sqrt{\pi}\,z^{-2m}}{\Gamma(\frac{1}{2}+m)}\\ &\times\int_0^{\frac{1}{2}\pi} (z^2+\cos^2\varphi)^{m-\frac{1}{2}}\,\mathbf{P}_n^m(\cos 2\varphi)\,\sin^{1-m}\varphi\,\cos^m\varphi\,d\varphi.\end{aligned}\right\} \quad . \quad . \quad (11)$$

Hierin wird $z \neq 0$, $|\arg z| < \frac{1}{2}\pi$ und $-\frac{1}{2} < \Re(m) < 1$ vorausgesetzt.

---

7) Ich benutze die Beziehung $\Gamma(\frac{1}{2}-m)\,\Gamma(1-m) = 2^{2m}\sqrt{\pi}\,\Gamma(1-2m)$.

8) HOBSON, [1], 210, Formel (41).

Aus (6) und (2) ergibt sich nämlich

$$U_n^m(\sqrt{1+z^2})\,U_{-n-1}^m(\sqrt{1+z^2}) = \frac{\sqrt{\pi}}{2\,z\,\Gamma(\frac{1}{2}+m)\,\Gamma(1-m)}$$

$$\times\int_0^1 (1+z^{-2}u)^{m-\frac{1}{2}}\,{}_2F_1(-n,1+n;1-m;1-u)\,(1-u)^{-m}\,u^{m-\frac{1}{2}}\,du;$$

diese Beziehung geht für $u=\cos^2\varphi$ über in

$$\left.\begin{aligned} U_n^m(\sqrt{1+z^2})\,U_{-n-1}^m(\sqrt{1+z^2}) &= \frac{\sqrt{\pi}\,z^{-2m}}{\Gamma(\frac{1}{2}+m)\,\Gamma(1-m)} \\ \times\int_0^{\frac{1}{2}\pi}(z^2+\cos^2\varphi)^{m-\frac{1}{2}}\,{}_2F_1&(-n,1+n;1-m;\sin^2\varphi)\sin^{1-2m}\varphi\cos^{2m}\varphi\,d\varphi. \end{aligned}\right\} \quad (12)$$

Wegen (1) gilt aber

$${}_2F_1(-n,1+n;1-m;\sin^2\varphi) = \Gamma(1-m)\sin^m\varphi\cos^{-m}\varphi\,\mathbf{P}_n^m(\cos 2\varphi).$$

Formel (11) folgt also sofort aus (12).

Genau so wie (10) und (11) beweist man, ausgehend von (4) bezw. (8),

$$\left.\begin{aligned} P_n^m(\sqrt{1+z^2})\,P_n^{m-1}(\sqrt{1+z^2}) &= \frac{2\,z^{1-2m}\sqrt{1+z^2}\;\Gamma(\frac{3}{2}-m)}{\sqrt{\pi}\;\Gamma(1-m-n)\,\Gamma(2-m+n)} \\ \times\int_0^\infty (z^2+\cosh^2 t)^{m-\frac{3}{2}}\,P_n^m&(\cosh 2t)\sinh^{1-m}t\cosh^m t\,dt \end{aligned}\right\} \quad (13)$$

(wo $z\neq 0$, $|\arg z|<\frac{1}{2}\pi$, $\Re(m+n)<1$, $\Re(m-n)<2$ und $\Re(m)<1$ ist) bezw.

$$\left.\begin{aligned} D_n^m(\sqrt{1+z^2}) &= \frac{\sqrt{\pi}\,z^{1-2m}\sqrt{1+z^2}}{\Gamma(m-\frac{1}{2})} \\ \times\int_0^{\frac{1}{2}\pi}(z^2+\cos^2\varphi)^{m-\frac{3}{2}}\,&\mathbf{P}_n^m(\cos 2\varphi)\sin^{1-m}\varphi\cos^m\varphi\,d\varphi \end{aligned}\right\} . \quad (14)$$

(wo $z\neq 0$, $|\arg z|<\frac{1}{2}\pi$ und $-\frac{1}{2}<\Re(m)<1$ ist).

Die entsprechenden Integraldarstellungen für die Funktionen $P_n^m(\sqrt{1+z^2})\,P_{n-1}^m(\sqrt{1+z^2})$ und $C_n^m(\sqrt{1+z^2})$ sind etwas komplizierter; es gilt nämlich

$$\left.\begin{aligned} P_n^m(\sqrt{1+z^2})\,P_{n-1}^m(\sqrt{1+z^2}) &= \frac{z^{-2m}\sqrt{1+z^2}\,\Gamma(\frac{1}{2}-m)}{\sqrt{\pi}\,\Gamma(1-m-n)\,\Gamma(1-m+n)}\int_0^\infty (z^2+\cosh^2 t)^{m-\frac{1}{2}} \\ \times\{(n-m)\,P_n^m(\cosh 2t)-(n+m)\,&P_{n-1}^m(\cosh 2t)\}\sinh^{-1-m}t\cosh^m t\,dt \end{aligned}\right\} . \quad (15)$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{2}\pi$, $\Re(m \pm n) < 1$ und $\Re(m) < 0$ ist) und

$$\left.\begin{array}{l} C_n^m(\sqrt{1+z^2}) = \dfrac{\sqrt{\pi}\, z^{-2m}\sqrt{1+z^2}}{2\,\Gamma(\frac{1}{2}+m)} \displaystyle\int_0^{\frac{1}{2}\pi} (z^2+\cos^2\varphi)^{m-\frac{1}{2}} \\ \times \{(n-m)\,\mathbf{P}_n^m(\cos 2\varphi) - (n+m)\,\mathbf{P}_{n-1}^m(\cos 2\varphi)\} \sin^{-1-m}\varphi \cos^m\varphi\, d\varphi \end{array}\right\}. \quad (16)$$

(wo $z \neq 0$, $|\arg z| < \frac{1}{2}\pi$ und $-\frac{1}{2} < \Re(m) < 0$ ist).

Diese Relationen folgen mit Hilfe von (2) aus (5) bezw. (7). Beim Beweis von (15) benutze ich die Hilfsformel

$$2\,m\,.\,{}_2F_1(-n, -m-n; -m; x) = (m-n)\,.\,{}_2F_1(-n, -m-n; 1-m; x)$$
$$+ (m+n)(1-x)\,.\,{}_2F_1(1-n, 1-m-n; 1-m; x);$$

ebenso bei der Ableitung von (16) die Hilfsformel

$$2\,m\,.\,{}_2F_1(-n, n; -m; x)$$
$$= (m-n)\,.\,{}_2F_1(-n, 1+n; 1-m; x) + (m+n)\,.\,{}_2F_1(1-n, n; 1-m; x).$$

Die Integraldarstellungen (10), (11), (13), (14), (15) und (16) sind sehr nahe verwandt mit den Beziehungen (35), . . . , (46) meiner Arbeit [3]. Z.B.: Formel (35) der angeführten Arbeit bekommt nach einiger Transformation die Gestalt

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{2\,z^{2m}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-m)}\cdot$$
$$\times \int_0^{\operatorname{arcsinh} z} (z^2 - \sinh^2 t)^{-m-\frac{1}{2}}\, P_n^m(\cosh 2t) \cosh^{1+m} t \sinh^{-m} t\, dt.$$

§ 3. Wie ich in einer vorigen Arbeit gezeigt habe, besitzt die Funktion $\{P_n^m(\sqrt{1+z^2})\}^2$ unter gewissen Voraussetzungen die Integraldarstellungen [9])

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{4}{\Gamma(-m-n)\,\Gamma(1-m+n)} \int_0^{\infty e^{-i\arg z}} K_{2n+1}(2v)\, J_{-m}^2(zv)\, dv \quad (17)$$

und

$$\{P_n^m(\sqrt{1+z^2})\}^2 = \frac{4}{\pi\,\Gamma(-m-n)\,\Gamma(1-m+n)} \int_0^{\infty e^{-i\arg z}} J_{2m}(2zv)\, K_{n+\frac{1}{2}}^2(v)\, dv. \quad (18)$$

---

9) Man vergl. [3], Formeln (11) und (23); ausser (17) und (18) gebe ich in [3] noch verschiedene mit (17) und (18) verwandte Formeln.

Ich werde jetzt einige verwandte Relationen ableiten. Zunächst beweise ich

$$U_n^m(\sqrt{1+z^2})\, U_{-n-1}^m(\sqrt{1+z^2}) = \frac{\pi}{4i}\int_L I_{2n+1}(2v)\, H_m^{(1)}(zv)\, H_m^{(2)}(zv)\, dv. \quad (19)$$

Diese Beziehung gilt für $z \neq 0$ mit $|\arg z| < \frac{1}{2}\pi$ und alle Werte von m und n; der Integrationsweg L läuft von $\infty e^{-\frac{1}{2}\pi i}$ nach $\infty e^{\frac{1}{2}\pi i}$ und vermeidet den Punkt $v = 0$ durch einen auf der rechten Seite der imaginären Achse liegenden Halbkreis.

Beweis von (19): Ich brauche nur den Fall mit $|\Re(m)| < \frac{1}{2}$ zu betrachten. Das Produkt $H_m^{(1)}(\zeta)\, H_m^{(2)}(\zeta)$ besitzt dann für $|\arg\zeta| < \pi$ die Integraldarstellung [10])

$$\left.\begin{aligned} H_m^{(1)}(\zeta)\, H_m^{(2)}(\zeta) = \frac{\cos m\pi}{\pi^{\frac{7}{2}} i}\int_{-\infty i+\sigma}^{\infty i+\sigma} \Gamma(\tfrac{1}{2}+s)\,\Gamma(-m-s)\,\Gamma(m-s)\,\Gamma(-s)\,\zeta^{2s}\, ds \\ [-\tfrac{1}{2} < \sigma < -|\Re(m)|]. \end{aligned}\right\} \quad (20)$$

Hieraus und aus [11])

$$\int_L I_\nu(2v)\, v^\lambda\, dv = \frac{\pi i}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\lambda)\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\lambda)} \qquad [\Re(\lambda) < \tfrac{1}{2}]$$

folgt, dass die rechte Seite von (19) gleich [12])

$$-\frac{\cos m\pi}{4\pi^{\frac{5}{2}}}\int_L I_{2n+1}(2v)\, dv \int_{-\infty i+\sigma}^{\infty i+\sigma} \Gamma(\tfrac{1}{2}+s)\,\Gamma(-m-s)\,\Gamma(m-s)\,\Gamma(-s)\,(zv)^{2s}\, ds$$

$$= -\frac{\cos m\pi}{4\pi^{\frac{5}{2}}}\int_{-\infty i+\sigma}^{\infty i+\sigma} \Gamma(\tfrac{1}{2}+s)\,\Gamma(-m-s)\,\Gamma(m-s)\,\Gamma(-s)\, z^{2s}\, ds \int_L I_{2n+1}(2v)\, v^{2s}\, dv$$

$$= \frac{\cos m\pi}{4\pi^{\frac{3}{2}} i}\int_{-\infty i+\sigma}^{\infty i+\sigma} \frac{\Gamma(\frac{1}{2}+s)\,\Gamma(-m-s)\,\Gamma(m-s)\,\Gamma(-s)}{\Gamma(-n-s)\,\Gamma(1+n-s)}\, z^{2s}\, ds$$

ist.

---

10) Man vergl. WATSON, [7], 223, Formel (2) mit $\mu = \nu = m$.

11) Siehe [6], Formel (3).

12) Nimmt man $\sigma < -\frac{1}{4}$, so ist die Vertauschung der Integrationsfolge erlaubt.

Das letzte Integral ist nach der BARNESschen Theorie der hypergeometrischen Funktionen gleich [13])

$$\frac{\pi}{2z\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{3}{2}+n)}\;{}_3F_2\left(\begin{matrix}\frac{1}{2}-m,\,\frac{1}{2}+m,\,\frac{1}{2};\\ \frac{1}{2}-n,\,\frac{3}{2}+n;\,-z^{-2}\end{matrix}\right).$$

Aus (6) ergibt sich aber, dass dieser Ausdruck den Wert

$$U_n^m\left(\sqrt{1+z^2}\right)\,U_{-n-1}^m\left(\sqrt{1+z^2}\right)$$

besitzt, so dass der Beweis von (19) geliefert is.

Auf analoge Weise wie (19) findet man (man vergl. (7))

$$C_n^m\left(\sqrt{1+z^2}\right)=\frac{\pi\sqrt{1+z^2}}{4i}\int\limits_L I_{2n}(2v)\,H_m^{(1)}(zv)\,H_m^{(2)}(zv)\,v\,dv\,. \quad . \qquad (21)$$

Die entsprechende Integraldarstellung der Funktion $D_n^m\left(\sqrt{1+z^2}\right)$ ist

$$\left.\begin{aligned}D_n^m\left(\sqrt{1+z^2}\right)=\frac{\pi\sqrt{1+z^2}}{8i}\int\limits_L I_{2n+1}(2v)\\ \times\{H_m^{(1)}(zv)\,H_{m-1}^{(2)}(zv)+H_m^{(2)}(zv)\,H_{m-1}^{(1)}(zv)\}\,v\,dv\,.\end{aligned}\right\}\quad .\quad .\quad .\qquad (22)$$

Diese Beziehung gilt, ebenso wie (21), für $z\neq 0$, $|\arg z|<\frac{1}{2}\pi$ und alle Werte von m und n.

Der Beweis von (22) ist dem von (19) analog; statt (20) und (6) benutze man

$$H_m^{(1)}(\zeta)\,H_{m-1}^{(2)}(\zeta)+H_m^{(2)}(\zeta)\,H_{m-1}^{(1)}(\zeta)$$

$$=\frac{2i\cos m\pi}{\pi^2}\int\limits_{-\infty i+\sigma}^{\infty i+\sigma}\Gamma(1+s)\,\Gamma(\tfrac{1}{2}-m-s)\,\Gamma(m-\tfrac{1}{2}-s)\,\Gamma(-\tfrac{1}{2}-s)\,\zeta^{2s}\,ds$$

und Entwicklung (8) [14]).

§ 4. Mit Hilfe der WHIPPLEschen Transformation [15])

$$P_n^m\left(\sqrt{1+z^2}\right)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}U_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}\left(\sqrt{1+z^{-2}}\right).\quad .\quad .\quad .\qquad (23)$$

---

13) Ich berechne die Summe der Residuen des Integranden in den Polen $s=-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$

14) In einer vorigen Arbeit habe ich (17), (18) und die mit (17) und (18) verwandten. Formeln (man vergl. Fussnote [9])) auf elementare Weise abgeleitet. Diese Formeln können aber auch auf analoge Weise wie (19), (21) und (22), nämlich mit Hilfe von BARNESschen Integralen, bewiesen werden.

15) WHIPPLE [**8**], 304; HOBSON, [**1**], 245—247; MEIJER, [**3**], 227.

kann man aus den Beziehungen (10), (11), (13), (14), (15), (16), (19), (21) und (22) noch andere Integraldarstellungen für Produkte von LEGENDREschen Funktionen ableiten. Aus (23) und (19) folgt z.B. [16])

$$P_n^m(\sqrt{1+z^2})\, P_n^{-m}(\sqrt{1+z^2}) = \frac{1}{2i}\int_M I_{2m}(2zv)\, H^{(1)}_{n+\frac{1}{2}}(v)\, H^{(2)}_{n+\frac{1}{2}}(v)\, dv. \quad (24)$$

Hierin ist $z \neq 0$ und $|\arg z| < \frac{1}{2}\pi$; der Integrationsweg M läuft von $\infty e^{-i(\frac{1}{2}\pi+\arg z)}$ nach $\infty e^{i(\frac{1}{2}\pi-\arg z)}$ und lässt den Punkt $v=0$ zur Linken.

Formel (24) entspricht (18).

LITERATURVERZEICHNIS.

E. W. HOBSON.

1. The theory of spherical and ellipsoidal harmonics (1931).

C. S. MEIJER.

2. Neue Integraldarstellungen aus der Theorie der WHITTAKERschen und HANKELschen Funktionen. Math. Annalen, **112**, 469—489 (1936).
3. Integraldarstellungen für Produkte von LEGENDREschen Funktionen. Nieuw Archief voor Wiskunde, (2) **19**, 207—234 (1938).
4. Einige Integraldarstellungen aus der Theorie der BESSELschen und WHITTAKERschen Funktionen. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 394—403 und 519—527 (1936).
5. Zur Theorie der hypergeometrischen Funktionen. Ibidem, **42**, 355—369 (1939).
6. Ueber BESSELsche, LOMMELsche und WHITTAKERsche Funktionen. (Erste Mitteilung). Ibidem, **42**, 872—879 (1939).

G. N. WATSON.

7. A treatise on the theory of BESSEL functions (1922).

F. J. W. WHIPPLE.

8. A symmetrical relation between LEGENDRE's functions with parameters cosh α and coth α. Proc. London Math. Soc., (2) **16**, 301—314 (1917).

[16]) Ich benutze die Beziehung $H^{(1)}_{-n-\frac{1}{2}} H^{(2)}_{-n-\frac{1}{2}} = H^{(1)}_{n+\frac{1}{2}} H^{(2)}_{n+\frac{1}{2}}$; man vergl. WATSON, [7], § 3.61, Formel (7).

Mathematics. — *Ueber* BESSEL*sche*, LOMMEL*sche und* WHITTAKER*sche Funktionen*. (Zweite Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of November 25, 1939.)

§ 5. In einer früheren Arbeit [18]) habe ich für das Produkt $J_\mu(z)\, J_\nu(z)$ die folgenden Integraldarstellungen abgeleitet

$$J_\mu(z)\, J_\nu(z) = \frac{2^{3-\mu-\nu}}{\pi} \int\limits_0^{\frac{1}{2}\pi} r_{\mu+\nu-1,\mu-\nu}(2z\cos\varphi)\cos(\mu+\nu)\varphi\, d\varphi \qquad [\Re(\mu+\nu) > -1] \tag{28}$$

und [19])

$$J_\mu(z)\, J_\nu(z) = \frac{2^{-3\mu-\nu+\frac{5}{2}}\, z^{-\mu}}{\sqrt{\pi}} \int\limits_0^{\frac{1}{2}\pi} r_{2\mu+\nu-1,\nu}(2z\cos\varphi)\, \mathbf{P}_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\varphi)\sin^{\frac{1}{2}-\mu}\varphi\cos^{-\mu}\varphi\, d\varphi \qquad [\Re(\mu) < \tfrac{1}{2},\ \Re(\mu+\nu) > -1]. \tag{29}$$

Die hierin auftretende Funktion $r_{\mu,\nu}(z)$ wird erklärt durch

$$\begin{aligned} r_{\mu,\nu}(z) &= \frac{s_{\mu,\nu}(z)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)} \\ &= \frac{z^{\mu+1}}{4\Gamma(\frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)}\ {}_1F_2(1;\tfrac{3}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\tfrac{3}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;-\tfrac{1}{4}z^2), \end{aligned} \tag{30}$$

wo $s_{\mu,\nu}(z)$ die erste LOMMELsche Funktion bezeichnet.

---

[18]) MEIJER, [11], 363—364.

[19]) Für $-1 < w < 1$ wird die zugeordnete LEGENDREsche Funktion erster Art definiert durch (siehe HOBSON, [3], 227)

$$\mathbf{P}_n^m(w) = \frac{(1+w)^{\frac{1}{2}m}(1-w)^{-\frac{1}{2}m}}{\Gamma(1-m)}\ {}_2F_1(-n, 1+n; 1-m; \tfrac{1}{2}-\tfrac{1}{2}w).$$

Die Funktion $\mathbf{P}_n^m(w)$ ist — im Gegensatz zu der in § 3 (siehe Fussnote [10])) betrachteten Funktion $P_n^m(w)$ — eindeutig für $-1 < w < 1$.

Es ist naheliegend auch die Integrale

$$\int_0^{\frac{1}{2}\pi} S_{\mu+\nu-1,\mu-\nu}(2z\cos\varphi)\cos(\mu+\nu)\varphi\,d\varphi$$

und

$$\int_0^{\frac{1}{2}\pi} S_{2\mu+\nu-1,\nu}(2z\cos\varphi)\,\mathbf{P}_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\varphi)\sin^{\frac{1}{2}-\mu}\varphi\cos^{-\mu}\varphi\,d\varphi$$

zu betrachten; hierin bedeutet $S_{\mu,\nu}(z)$ die zweite LOMMELsche Funktion.

Ich setze

$$R_{\mu,\nu}(z)=\frac{S_{\mu,\nu}(z)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)} \quad . \quad . \quad . \quad (31)$$

und werde zeigen

$$\left.\begin{aligned}&\{H_\mu^{(1)}(z)\,H_\nu^{(2)}(z)-H_\nu^{(1)}(z)\,H_\mu^{(2)}(z)\}\sin\mu\pi\sin\nu\pi\\&=\frac{2^{4-\mu-\nu}\sin(\mu-\nu)\pi}{\pi i}\int_0^{\frac{1}{2}\pi}R_{\mu+\nu-1,\mu-\nu}(2z\cos\varphi)\cos(\mu+\nu)\varphi\,d\varphi\end{aligned}\right\} \quad (32)$$

(wo $\Re(\mu+\nu)>-1$ und $|\Re(\mu-\nu)|<1$ ist) und

$$\left.\begin{aligned}&\{H_{-\mu}^{(1)}(z)\,H_\nu^{(2)}(z)-H_\nu^{(1)}(z)\,H_{-\mu}^{(2)}(z)\}\sin\mu\pi\\&=\frac{2^{-3\mu-\nu+\frac{7}{2}}\,i\,z^{-\mu}}{\sqrt{\pi}}\int_0^{\frac{1}{2}\pi}R_{2\mu+\nu-1,\nu}(2z\cos\varphi)\,\mathbf{P}_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\cos\varphi)\sin^{\frac{1}{2}-\mu}\varphi\cos^{-\mu}\varphi\,d\varphi\end{aligned}\right\} (33)$$

(wo $\Re(\mu)<\frac{1}{2}$, $\Re(\mu-\nu)<1$ und $|\Re(\mu+\nu)|<1$ ist).

Für die durch (31) definierte Funktion $R_{\mu,\nu}(z)$ gilt nämlich [20])

$$R_{\mu,\nu}(z)\sin\nu\pi=r_{\mu,\nu}(z)\sin\nu\pi+2^{\mu-1}\{J_{-\nu}(z)\cos\tfrac{1}{2}(\mu-\nu)\pi-J_\nu(z)\cos\tfrac{1}{2}(\mu+\nu)\pi\}; \ (34)$$

ferner hat man [21])

$$\left.\begin{aligned}J_\mu(z)\,J_\nu(z)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}J_{\mu+\nu}(2z\cos\varphi)\cos(\mu-\nu)\varphi\,d\varphi\\ [\Re(\mu+\nu)>-1]\end{aligned}\right\} \quad . \quad . \quad (35)$$

---

20) Man vergl. WATSON, [12], 347.

21) Siehe MEIJER, [11], 363.

und

$$J_\mu(z)\,J_\nu(z) = \frac{2^{\mu+\frac{1}{2}}\,z^\mu}{\sqrt{\pi}} \int_0^{\frac{1}{2}\pi} J_\nu(2\,z\cos\varphi)\,\mathbf{P}_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cos\varphi)\,\sin^{\mu+\frac{1}{2}}\varphi\,\cos^\mu\varphi\,d\varphi \quad [\Re(\mu) > -\tfrac{1}{2},\ \Re(\mu+\nu) > -1]. \qquad (36)$$

Nun folgt aus (34)

$$2^{3-\mu-\nu}\sin(\mu-\nu)\pi\,R_{\mu+\nu-1,\mu-\nu} = 2^{3-\mu-\nu}\sin(\mu-\nu)\pi\,r_{\mu+\nu-1,\mu-\nu} + 2\sin\nu\pi\,J_{-\mu+\nu} - 2\sin\mu\pi\,J_{\mu-\nu}.$$

Die rechte Seite von (32) ist also mit Rücksicht auf (28) und (35) gleich

$$-2\,i\sin(\mu-\nu)\pi\,J_\mu(z)\,J_\nu(z) - 2\,i\sin\nu\pi\,J_{-\mu}(z)\,J_\nu(z) + 2\,i\sin\mu\pi\,J_\mu(z)\,J_{-\nu}(z)$$

und dieser Ausdruck ist wegen [22])

$$H_\nu^{(1)}(z)\sin\nu\pi = i\,e^{-\nu\pi i}J_\nu(z) - i\,J_{-\nu}(z) \quad \text{und} \quad H_\nu^{(2)}(z)\sin\nu\pi = i\,J_{-\nu}(z) - i\,e^{\nu\pi i}\,J_\nu(z)$$

gleich der linken Seite von (32), so dass der Beweis von (32) geliefert ist.

Formel (33) folgt auf analoge Weise aus (29) und (36).

§ 6. Wendet man (1) auf die durch (30) definierte Funktion $r_{\mu,\nu}(z)$ an, so findet man (ich ersetze t durch $\frac{1}{2}t$ in (1))

$$r_{\mu,\nu}(z) = \frac{2^{\alpha+\beta-4}\,z^{\mu+1}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)\,\pi i}\int_{L_z} I_{\alpha-\beta}(t) \times {}_3F_2(\alpha,\beta,1;\tfrac{3}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\tfrac{3}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;-z^2/t^2)\,t^{-\alpha-\beta+1}\,dt. \qquad (37)$$

Eine entsprechende Integraldarstellung für die zweite LOMMELsche Funktion $S_{\mu,\nu}(z)$ habe ich früher [23]) schon abgeleitet.

Setzt man $\alpha = \frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu$ und $\beta = \frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu$ in (37), so bekommt man die besonders einfache Formel

$$r_{\mu,\nu}(z) = \frac{2^{\mu-1}\,z^{\mu+1}}{\pi i}\int_{L_z}\frac{I_\nu(t)\,t^{-\mu}\,dt}{t^2+z^2} \qquad [\Re(\mu) > -\tfrac{5}{2}]. \qquad (38)$$

Mit Hilfe von (38) beweist man leicht [24])

$$\int_0^\infty r_{\mu,\nu}(2\,u)\,u^{\lambda-1}\,du = \frac{2^{\mu-2}\,\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\lambda-\frac{1}{2}\mu)}{\Gamma(1-\frac{1}{2}\lambda+\frac{1}{2}\nu)\,\Gamma(1-\frac{1}{2}\lambda-\frac{1}{2}\nu)}; \qquad (39)$$

---

22) WATSON, [12], 74.

23) MEIJER, [5], Formel (12).

24) Formel (39) kann auch mit Hilfe der MELLINschen Umkehrformel abgeleitet werden.

diese Beziehung gilt für

$$\Re(\lambda) - 1 < -\Re(\mu) < \Re(\lambda) + 1 < \tfrac{5}{2} \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

Aus (30) und (34) folgt nämlich mit Rücksicht auf die asymptotischen Entwicklungen der Funktionen [25] $R_{\mu,\nu}(z)$ und $J_\nu(z)$, dass die linke Seite von (39) unter der Voraussetzung (40) konvergiert. Nach der Theorie der analytischen Fortsetzung brauche ich also nur den Fall mit

$$\Re(\lambda) - 1 < -\Re(\mu) < \Re(\lambda) + 1 < \tfrac{3}{2} \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

zu betrachten. Die linke Seite von (39) ist dann infolge (38) gleich [26]

$$\frac{2^{\mu-1}}{\pi i} \int_0^\infty u^{\lambda+\mu}\, du \int_{L_u} \frac{I_\nu(2t)\, t^{-\mu}\, dt}{t^2 + u^2}$$

$$= \frac{2^{\mu-1}}{\pi i} \int_{L_u} I_\nu(2t)\, t^{-\mu}\, dt \int_0^\infty \frac{u^{\lambda+\mu}\, du}{t^2 + u^2}$$

$$= \frac{2^{\mu-2}\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\lambda + \tfrac{1}{2}\mu)\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\lambda - \tfrac{1}{2}\mu)}{\pi i} \int_{L_u} I_\nu(2t)\, t^{\lambda-1}\, dt$$

$$= \frac{2^{\mu-2}\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\lambda + \tfrac{1}{2}\mu)\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}\lambda - \tfrac{1}{2}\mu)}{\Gamma(1 - \tfrac{1}{2}\lambda + \tfrac{1}{2}\nu)\, \Gamma(1 - \tfrac{1}{2}\lambda - \tfrac{1}{2}\nu)} \qquad \text{(wegen (3))}.$$

Hiermit ist (39) bewiesen.

§ 7. Eine mit (28) und (32) verwandte Beziehung ist

$$I_{-\mu}(z)\, I_{-\nu}(z) - I_\mu(z)\, I_\nu(z) = \frac{2^{3-\mu-\nu} \sin(\mu+\nu)\pi}{\pi} \int_0^\infty r_{\mu+\nu-1,\,\mu-\nu}(2z\sinh t)\, e^{(\mu+\nu)t}\, dt; \qquad (42)$$

hierin wird $z > 0$ und $|\Re(\mu+\nu)| < 1$ vorausgesetzt.

Aus (8) ergibt sich nämlich

$${}_2F_1(\tfrac{1}{2} + a, 1 + a; 1 + 2a; -\operatorname{cosech}^2 t) = 2^{2a}\, e^{-2at} \sinh^{2a+1} t \cosh^{-1} t.$$

Folglich ist

$$e^{(\mu+\nu)t} = 2^{\mu+\nu} \sinh^{\mu+\nu-1} t \cosh t \,.\, {}_2F_1\left(\begin{matrix} \tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\nu,\ 1 - \tfrac{1}{2}\mu - \tfrac{1}{2}\nu; \\ 1 - \mu - \nu; -\operatorname{cosech}^2 t \end{matrix}\right).$$

[25]) Für die asymptotischen Entwicklungen der Funktionen $R_{\mu,\nu}(z)$ und $J_\nu(z)$ siehe man WATSON, [12], 351 und 199.

[26]) Die Vertauschung der Integrationsfolge ist wegen (41) erlaubt.

Die rechte Seite von (42) ist daher gleich [27])

$$\frac{8\,z^{-\mu-\nu}\sin(\mu+\nu)\pi}{\pi}\int_0^\infty r_{\mu+\nu-1,\,\mu-\nu}(2u)\,.\,{}_2F_1\left(\begin{matrix}\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu,\,1-\frac{1}{2}\mu-\frac{1}{2}\nu;\\ 1-\mu-\nu;\,-z^2/u^2\end{matrix}\right)u^{\mu+\nu-1}\,du. \quad (43)$$

Aus (30), (34) und dem Verhalten der Funktionen [28]) $R_{\mu,\nu}(u)$ und $J_\nu(u)$ für $u\to\infty$ folgt, dass das Integral (43) konvergiert für $|\Re(\mu+\nu)|<1$. Ich werde zeigen, dass dieses Integral gleich der linken Seite von (42) ist. Nach der Theorie der analytischen Fortsetzung darf ich hierbei $|\Re(\mu+\nu)|<\frac{1}{2}$ annehmen. Es existiert dann eine reelle Zahl σ mit

$$\Re\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu-\tfrac{1}{4}\right)<\sigma<\mathrm{Min}\,\{0,\Re(\mu+\nu)\}\,.\quad .\quad .\quad .\quad (44)$$

Nun besitzt die in (43) vorkommende hypergeometrische Funktion ${}_2F_1(-z^2/u^2)$ die Integraldarstellung [29])

$${}_2F_1(-z^2/u^2)=\frac{2^{-\mu-\nu-1}}{\pi^{\frac{3}{2}}\,i}\int_{-\infty i+\sigma}^{\infty i+\sigma}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu+s)\,\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu+s)\,\Gamma(-s)}{\Gamma(1-\mu-\nu+s)}\left(\frac{z^2}{u^2}\right)^s ds. \quad (45)$$

Für jedes s mit $\Re(s)=\sigma$ gilt ferner wegen (44)

$$\Re(\mu+\nu-1)<\Re(s)<\Re(\mu+\nu)\ \text{ und }\ \Re(s)>\Re\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu-\tfrac{1}{4}\right),$$

so dass Formel (39) mit $\mu+\nu-1$ statt μ, $\mu-\nu$ statt ν und $\mu+\nu-2s$ statt λ angewendet werden darf; man findet

$$\int_0^\infty r_{\mu+\nu-1,\,\mu-\nu}(2u)\,u^{\mu+\nu-2s-1}\,du=\frac{2^{\mu+\nu-3}\,\Gamma(\mu+\nu-s)\,\Gamma(1-\mu-\nu+s)}{\Gamma(1-\nu+s)\,\Gamma(1-\mu+s)}\,.\,. \quad (46)$$

Setzt man nun in (43) für ${}_2F_1(-z^2/u^2)$ das Integral (45) ein, so erhält man, wenn man die Integrationsfolge vertauscht [30]) und Beziehung (46) benutzt,

$$\frac{z^{-\mu-\nu}\sin(\mu+\nu)\pi}{2\pi^{\frac{5}{2}}\,i}\int_{-\infty i+\sigma}^{\infty i+\sigma}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu+s)\,\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu+s)\,\Gamma(-s)\,\Gamma(\mu+\nu-s)}{\Gamma(1-\nu+s)\,\Gamma(1-\mu+s)}\,z^{2s}\,ds.$$

---

27) Ich ersetze sinh t durch u/z.

28) WATSON, [12], 351 und 199.

29) Ich berechne die Summe der Residuen des Integranden in den Polen $s=0,1,2,\ldots$; diese Pole liegen wegen (44) auf der rechten, die Pole $\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}$, $\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{3}{2}$, $\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{5}{2},\ldots$ und $\frac{1}{2}\mu+\frac{1}{2}\nu-1$, $\frac{1}{2}\mu+\frac{1}{2}\nu-2$, $\frac{1}{2}\mu+\frac{1}{2}\nu-3,\ldots$ aber auf der linken Seite des Integrationsweges.

30) Die Vertauschung der Integrationsfolge ist erlaubt wegen $\Re(s)>\Re\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{4}\right)$.

Dieser Ausdruck ist gleich [31])

$$\frac{2^{\mu+\nu}\, z^{-\mu-\nu}}{\Gamma(1-\mu)\,\Gamma(1-\nu)}\, {}_2F_3\left(\begin{matrix}\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu,\ 1-\frac{1}{2}\mu-\frac{1}{2}\nu;\\ 1-\mu,\ 1-\nu,\ 1-\mu-\nu;\ z^2\end{matrix}\right)$$

$$-\frac{2^{-\mu-\nu}\, z^{\mu+\nu}}{\Gamma(1+\mu)\,\Gamma(1+\nu)}\, {}_2F_3\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu,\ 1+\frac{1}{2}\mu+\frac{1}{2}\nu;\\ 1+\mu,\ 1+\nu,\ 1+\mu+\nu;\ z^2\end{matrix}\right)$$

$$= I_{-\mu}(z)\, I_{-\nu}(z) - I_{\mu}(z)\, I_{\nu}(z),$$

womit der Beweis von (42) geliefert ist.

§ 8. Neuerdings [32]) habe ich unter gewissen Voraussetzungen für das Produkt $M_{k,m}(z)\, M_{-k,m}(z)$ die folgende Integraldarstellung abgeleitet [33])

$$\left.\begin{aligned} M_{k,m}(z)\, M_{-k,m}(z) &= \frac{2^{-m+\frac{1}{2}}\, z^{m+\frac{1}{2}}\, \Gamma^2(1+2m)\sqrt{\pi}}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-k+m)} \\ \times \int_0^\infty I_{m-\frac{1}{2}}(z \operatorname{sech} v)\, & P_{k-\frac{1}{2}}^{-m}(\cosh 2v) \sinh^{m+1} v \cosh^{-2m-\frac{3}{2}} v\, dv. \end{aligned}\right\} \quad (47)$$

Ich werde jetzt einige entsprechende Beziehungen angeben.
Ist $\Re(k) < 1$, $\Re(k+m) < \frac{1}{2}$ und $|\arg z| < \frac{1}{2}\pi$, so gilt nämlich [34])

$$\left.\begin{aligned} W_{k,m}(iz)\, W_{k,m}(-iz) &= \frac{2^{2k+m+\frac{1}{2}}\, z^{2k}}{\Gamma(1-k)\,\Gamma(\frac{1}{2}-k-m)} \int_0^{\infty e^{i\arg z}} K_{m+\frac{1}{2}}(u) \\ \times\, {}_2F_1(\tfrac{1}{2}-k+m,\ &\tfrac{1}{2}-k;\ 1-2k;\ -u^2/z^2)\, u^{-2k-m+\frac{1}{2}}\, du. \end{aligned}\right\} \quad (48)$$

Weiter hat man [35])

$$\left.\begin{aligned} W_{k,m}(z)\, M_{-k,m}(z) &= \frac{2^{2k+m-\frac{1}{2}}\, z^{2k}\, \Gamma(1+2m)}{\Gamma(1-k)} \int_0^\infty J_{m+\frac{1}{2}}(u) \\ \times\, {}_2F_1(\tfrac{1}{2}-k+m,\ &\tfrac{1}{2}-k;\ 1-2k;\ -u^2/z^2)\, u^{-2k-m+\frac{1}{2}}\, du; \end{aligned}\right\} \quad (49)$$

---

31) Ich berechne die Summe der Residuen des Integranden in den Polen $s = 0, 1, 2, \ldots$ und $s = \mu+\nu,\ \mu+\nu+1,\ \mu+\nu+2, \ldots$

32) MEIJER, [11], Formel (20).

33) Für die Definition der Funktion $P_n^m(w)$ siehe man Fussnote [10]).

34) MEIJER, [5], Formel (13) mit $\alpha = 1-k$, $\beta = \frac{1}{2}-k-m$ und $\tau = \arg z$.

35) MEIJER, [9], Formel (18) mit $\alpha = 1+m$ und $\beta = \frac{1}{2}$.

hierin ist $\Re(k) < 1$, $\Re(m) > -\frac{1}{3}$ und $|\arg z| < \frac{1}{2}\pi$. Ausserdem gilt [36])

$$\left.\begin{aligned} W_{k,m}(z)\{W_{k,m}(z e^{\pi i}) - W_{k,m}(z e^{-\pi i})\} = \frac{2^{2k+m+\frac{1}{2}} z^{2k} \pi i}{\Gamma(1-k)\,\Gamma(\frac{1}{2}-k-m)} \int_0^\infty J_{-m-\frac{1}{2}}(u) \\ \times {}_2F_1(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k; 1-2k; -u^2/z^2)\, u^{2k-m+\frac{1}{2}}\, du; \end{aligned}\right\} \quad (50)$$

hierin ist $\Re(k+m) < \frac{1}{2}$, $\Re(m) > -\frac{1}{3}$ und $|\arg z| < \frac{1}{2}\pi$.

Nun ist [37])

$$\frac{1}{\Gamma(1-k)}\, {}_2F_1(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k; 1-2k; -\operatorname{cosech}^2 v)$$

$$= \frac{2^{1-2k} e^{m\pi i}}{\Gamma(\frac{1}{2}-k-m)\sqrt{\pi}} \cosh^{-m} v \sinh^{1-2k+m} v\, Q^{-m}_{-k-\frac{1}{2}}(\cosh 2v),$$

wo $Q^{\mu}_{\nu}(\zeta)$ die zugeordnete LEGENDREsche Funktion zweiter Art bezeichnet [38]).

Aus (48) mit $u = z \operatorname{cosech} v$ ergibt sich somit

$$\left.\begin{aligned} W_{k,m}(iz)\, W_{k,m}(-iz) = \frac{2^{m+\frac{1}{2}} z^{-m+\frac{3}{2}} e^{m\pi i}}{\Gamma^2(\frac{1}{2}-k-m)\sqrt{\pi}} \\ \times \int_0^\infty K_{m+\frac{1}{2}}(z \operatorname{cosech} v)\, Q^{-m}_{-k-\frac{1}{2}}(\cosh 2v) \cosh^{-m+1} v \sinh^{2m-\frac{3}{2}} v\, dv. \end{aligned}\right\} \quad (51)$$

Auf analoge Weise liefern (49) und (50), falls $z > 0$ ist,

$$\left.\begin{aligned} W_{k,m}(z)\, M_{-k,m}(z) = \frac{2^{m+\frac{1}{2}} z^{-m+\frac{3}{2}} e^{m\pi i}\, \Gamma(1+2m)}{\Gamma(\frac{1}{2}-k-m)\sqrt{\pi}} \\ \times \int_0^\infty J_{m+\frac{1}{2}}(z \operatorname{cosech} v)\, Q^{-m}_{-k-\frac{1}{2}}(\cosh 2v) \cosh^{-m+1} v \sinh^{2m-\frac{3}{2}} v\, dv \end{aligned}\right\} \quad (52)$$

und

$$\left.\begin{aligned} W_{k,m}(z)\{W_{k,m}(z e^{\pi i}) - W_{k,m}(z e^{-\pi i})\} = \frac{2^{m+\frac{3}{2}} z^{-m+\frac{3}{2}} e^{m\pi i}\, i\sqrt{\pi}}{\Gamma^2(\frac{1}{2}-k-m)} \\ \times \int_0^\infty J_{-m-\frac{1}{2}}(z \operatorname{cosech} v)\, Q^{-m}_{-k-\frac{1}{2}}(\cosh 2v) \cosh^{-m+1} v \sinh^{2m-\frac{3}{2}} v\, dv. \end{aligned}\right\} \quad (53)$$

Die Beziehungen (51), (52) und (53) entsprechen (47).

---

36) MEIJER, [9], Formel (21) mit $\alpha = \frac{1}{2}$ und $\beta = -m$.

37) Man vergl. HOBSON, [3], 203, Formel (28).

38) Ich benutze die HOBSONsche Definition der Funktion $Q^{\mu}_{\nu}(\zeta)$, nicht die BARNESsche man vergl. HOBSON, [3], 196.

Eine verwandte Integraldarstellung für $W_{k,m}(z)\,W_{-k,m}(z)$ lautet wie folgt [39])

$$\left.\begin{aligned} W_{k,m}(z)\,W_{-k,m}(z) = \frac{2^{m+\frac12}\,z^{-m+\frac32}}{\sqrt{\pi}} \int_0^{\frac12\pi} K_{m+\frac12}(z\sec\varphi) \\ \times \mathbf{P}_{k-\frac12}^{m}(\cos 2\varphi)\,\sin^{-m+1}\varphi\,\cos^{2m-\frac32}\varphi\,d\varphi\,; \end{aligned}\right\} \quad . \quad . \quad (54)$$

hierin ist $\Re(m) < 1$ und $|\arg z| < \frac12\pi$.

Beim Beweis von (54) gehe ich aus von [40])

$$\left.\begin{aligned} W_{k,m}(z)\,W_{-k,m}(z) = -\frac{2^{2k+m-\frac12}\,z^{-m+\frac32}\,\Gamma(\frac12-k+m)}{\Gamma(1-k)\,\pi i} \int_\infty^{(1+)} K_{m+\frac12}(zu) \\ \times {}_2F_1(\tfrac12-k+m,\tfrac12-k;1-2k;u^2)\,u^{-2k-m+\frac12}\,du. \end{aligned}\right\} \quad . \quad (55)$$

Für die hierin auftretende hypergeometrische Funktion ${}_2F_1$ gilt nach einer bekannten Transformationsformel [41])

$$\begin{aligned} &{}_2F_1(\tfrac12-k+m,\tfrac12-k;1-2k;u^2) \\ &= \frac{\Gamma(1-2k)\,\Gamma(-m)}{\Gamma(\frac12-k-m)\,\Gamma(\frac12-k)}\,u^{2k-1}\,{}_2F_1(\tfrac12-k,\tfrac12+k;1+m;1-1/u^2) \\ &+ \frac{\Gamma(1-2k)\,\Gamma(m)}{\Gamma(\frac12-k+m)\,\Gamma(\frac12-k)}\,(1-u^2)^{-m}\,u^{2k-1}\,{}_2F_1(\tfrac12-k,\tfrac12+k;1-m;1-1/u^2). \end{aligned}$$

Wegen [42])

$$\int_\infty^{(1+)} K_{m+\frac12}(zu)\,.\,{}_2F_1(\tfrac12-k,\tfrac12-k;1+m;1-1/u^2)\,u^{-m-\frac12}\,du = 0$$

geht (55) also in

$$\begin{aligned} W_{k,m}(z)\,W_{-k,m}(z) = -\frac{2^{m-\frac12}\,z^{-m+\frac32}\,\Gamma(m)}{\pi i\sqrt{\pi}} \int_\infty^{(1+)} K_{m+\frac12}(zu) \\ \times {}_2F_1(\tfrac12-k,\tfrac12+k;1-m;1-1/u^2)\,(1-u^2)^{-m}\,u^{-m-\frac32}\,du \end{aligned}$$

39) Für die Definition der Funktion $\mathbf{P}_n^m(w)$ vergl. man Fussnote [19]).

40) MEIJER, [7], Formel (4) mit $\alpha = 1-k$ und $\beta = \frac12-k-m$. Relation (55) gilt für $|\arg z| < \frac12\pi$.

41) BARNES, [1], 152, Formel (IX).

42) Der Integrand ist analytisch im Innern des Integrationsweges.

über. Diese Beziehung ist für $\Re(m) < 1$ gleichwertig mit

$$\left.\begin{aligned} W_{k,m}(z)\, W_{-k,m}(z) = \frac{2^{m+\frac{1}{2}}\, z^{-m+\frac{3}{2}}}{\Gamma(1-m)\sqrt{\pi}} \int_1^\infty K_{m+\frac{1}{2}}(z u) \\ \times {}_2F_1\left(\tfrac{1}{2}-k, \tfrac{1}{2}+k; 1-m; 1-1/u^2\right)(u^2-1)^{-m}\, u^{-m-\frac{1}{2}}\, du. \end{aligned}\right\} \quad (56)$$

Nun ist [43])

$${}_2F_1\left(\tfrac{1}{2}-k, \tfrac{1}{2}+k; 1-m; \sin^2\varphi\right) = \Gamma(1-m)\sin^m\varphi\cos^{-m}\varphi\; \mathbf{P}^m_{k-\frac{1}{2}}(\cos 2\varphi);$$

man findet also (54), wenn man $u = \sec\varphi$ setzt in (56).

§ 9. Das Produkt $M_{k,m}(iz)\, M_{k,m}(-iz)$ besitzt wegen (25) und (1) die Integraldarstellung (ich ersetze t durch $\frac{1}{2}t$ in (1))

$$\left.\begin{aligned} M_{k,m}(iz)\, M_{k,m}(-iz) = \frac{2^{\alpha+\beta-2}\, z^{2m+1}\, \Gamma(\alpha)\, \Gamma(\beta)}{\pi i} \int_{L_z} I_{\alpha-\beta}(t) \\ \times {}_4F_3\left(\alpha, \beta, \tfrac{1}{2}+m+k, \tfrac{1}{2}+m-k; \tfrac{1}{2}+m, 1+m, 1+2m; -z^2/t^2\right) t^{-\alpha-\beta+1}\, dt. \end{aligned}\right\} \quad (57)$$

Nimmt man nun $\alpha = \beta = \frac{1}{2} + m$, so erhält man mit Rücksicht auf (15)

$$M_{k,m}(iz)\, M_{k,m}(-iz) = \frac{z\, \Gamma^2(1+2m)}{2i} \int_{L_z} I_0(t) \left\{ P^{-m}_{k-\frac{1}{2}}\left(\sqrt{1+\frac{z^2}{t^2}}\right) \right\}^2 dt;$$

hierin ist $z > 0$ und $\Re(m) > -\frac{1}{4}$.

Verwandte Integraldarstellungen für $W_{k,m}(iz)\, W_{k,m}(-iz)$, $W_{k,m}(z)\, W_{-k,m}(z)$, $W_{k,m}(z)\, M_{-k,m}(z)$ u.s.w. waren schon bekannt [44]).

Setzt man $\alpha = 1 + 2m$ und $\beta = \frac{1}{2} + m$ in (57), so findet man

$$\left.\begin{aligned} M_{k,m}(iz)\, M_{k,m}(-iz) = \frac{2^{3m-\frac{1}{2}}\, z^{2m+1}\, \Gamma(1+2m)\, \Gamma(\frac{1}{2}+m)}{\pi i} \int_{L_z} I_{m+\frac{1}{2}}(t) \\ \times {}_2F_1\left(\tfrac{1}{2}+m+k, \tfrac{1}{2}+m-k; 1+m; -z^2/t^2\right) t^{-3m-\frac{1}{2}}\, dt \qquad [\Re(m) > -\tfrac{1}{3}]. \end{aligned}\right\} \quad (58)$$

Nun ist [45])

$$\begin{aligned} &{}_2F_1\left(\tfrac{1}{2}+m+k, \tfrac{1}{2}+m-k; 1+m; -\sinh^2 v\right) \\ &= \Gamma(1+m)\cosh^{-m} v\, \sinh^{-m} v\; P^{-m}_{k-\frac{1}{2}}(\cosh 2v). \end{aligned}$$

---

43) Man vergl. Fussnote [19]).

44) MEIJER, [7], 487; [8], 522; [9], 139—141.

45) HOBSON, [3], 190, Formel (13).

Aus (58) mit $t = z \operatorname{cosech} v$ folgt also (ich nehme an, dass $z > 0$ und $\Re(m) > -\frac{1}{3}$ ist)

$$\left.\begin{aligned} M_{k,m}(iz)\, M_{k,m}(-iz) = -\frac{2^{m-\frac{1}{2}}\, z^{-m+\frac{3}{2}}\, \Gamma^2(1+2m)}{i\sqrt{\pi}} \\ \int\limits_C I_{m+\frac{1}{2}}(z \operatorname{cosech} v)\, P_{k-\frac{1}{2}}^{-m}(\cosh 2v) \cosh^{-m+1} v \sinh^{2m-\frac{3}{2}} v\, dv; \end{aligned}\right\} \qquad (59)$$

der Integrationsweg C besteht aus der imaginären Achse von 0 bis ri $(0 < r < \frac{1}{2}\pi)$, dem auf der rechten Seite der imaginären Achse liegenden Halbkreis mit Radius r (von ri bis $-ri$ durchlaufen) und der imaginären Achse von $-ri$ bis 0.

Formel (59) ist mit (47), (51), (52), (53) und (54) verwandt.

Histology. — *The different structures of the cyto-architectonic fields of the cerebral cortex as different manifestations of a general scheme, each being mainly indicated by the value of one varying property, called the field exponent.* By S. T. Bok. (Communicated by Prof. M. W. Woerdeman).

(Communicated at the meeting of November 25, 1939.)

The microscopic appearance of the cytoarchitectonic fields of the cerebral neocortex suggests that all these fields are different manifestations of a common scheme of structure. The question arises whether the differences seen between the fields are due to changes of only one property of the underlying structural principle. In that case all the other properties of that principle would be constant over the whole neocortex and the structure of each special architectonic field would be determined by the special value of the variable one. The other possibility is, that the structural principle contains many variable properties changing independently of each other. In that case at the border between two special fields a change might occur in a number of these properties and at the border between two other fields changes might occur in another combination of properties.

In my paper "A quantitative analysis of the structure of the cerebral cortex" (Royal Academy of Science, Amsterdam, XXXV, 2, 1936) the results were described of a series of measurings of the size of the ganglion cells at the various distances from the pia mater in one field, the area temporalis superior posterior of man. The size of the nerve cells — expressed by the volume of their nucleus — proved to be dependent upon the depth below the pia in such a way, that a fairly simple scheme could be given of the mutual relations between these volumes and depths.

The above problem can be studied by executing the same type of measuring in other fields of the human neocortex: do the relations between nucleus volume and depth in these other fields correspond to analogic schemes and, if so, what are the differences between these schemes? Do they depend upon one property that has a different value in the different fields, or must we describe the differences between two special schemes as different values of the one property and those between two other schemes as the variation of another property?

By plotting the nucleus volumes and depths measured in the area temp. sup. post. into a rectangular scheme (Fig. 5 l.c.) a correlation diagram was obtained that in two aspects differs from those common in literature.

In the first place the relation points (each indicating the two values of one cell) are distributed in two fields, called the upper and the lower main group. The upper group is built up by the nerve cells of the 2d and 3d layer and by some larger ones in the 4th layer, the lower group by the other (small) nerve cells of the 4th layer and those of the 5th and 6th. The two fields touch each other in a part where the density of the points is low. The boundary between them, in consequence, is rather sharp.

In the second place the density of the points in the different parts of each group is very unequal: at the side of the small values (of the nucleus volumes as well as of the depths) the points are found very close together, their density greatly decreases towards the opposite side of the field. If the logarithms of the values measured are plotted in the same way this dissymmetry disappears, however. This remarkable fact shows that the distribution of the values is not based upon an arithmetic progression (each nucleus volume presented being a constant amount larger than the former one in size) but upon a geometric progression (each volume presented being a constant number of times as large as the former in size).

By this peculiarity the correlation of the logarithms is easier to read than that of the values measured themselves. In each of the two groups it happens to be a simple one showing remarkable relations.

Both fields are oval in shape.

In the diagram shown in fig. 6 l.c., the volumes were plotted horizontally and the depths vertically. A number of horizontal straight lines at equal distances from each other was imagined, the mean of the *log* nucleus volumes between each pair of lines was reckoned and plotted as small circles (fig. 9 l.c.). In the upper zone these circles proved to be situated very near to a straight line with a gradient of 45° (tangent = 1): *in the upper zone the mean nucleus volume increases proportional to the depth.*

The nerve cells with a nucleus volume equal to the mean nucleus volume of their depth, thus, have the same quotient of their nucleus volume and depth. If the logarithms of these quotients are plotted against the logarithms of their depths, the relation points of these mean cells will be situated along a straight vertical line (small circlets in fig. 10 l.c.), so that one regression line in this new diagram is exactly vertical. The other regression line (through the mean depths of the cells with the same nucleus volume) is horizontal. *This means that the quotient nucleus volume divided by depth varies independently of the depth.*

In this diagram, moreover, the upper field is found to be a circle: its horizontal diameter is equal to its vertical diameter. The logarithms of the quotient, thus, vary as much as the logarithms of the depths. *This means that the quotient nucleus volume divided by depth varies as many times as the depths.*

The logarithms of these quotients show a normal distribution (frequen-

cy curve of GAUSS). The extension of the variations, thus, can be described accurately by one value, the standard variation (being equal to 0.155).

To express these relations more clearly a circle is drawn in the second diagram round the upper field. The borders of the field not being sharp, this circle has not a quite exact meaning, but nevertheless it illustrates the equality of the range of variation in the horizontal and vertical direction and it indicates in a simple way the practical extension of that variability. The oval drawn round the upper field in the first diagram (fig. 9 l.c., demonstrating the relation between *log* nucleus volume and *log* depth) is the transformation of this circle, and so is the curve round this field in the diagram of the nucleus volumes and depths themselves in fig. 5 l.c.

The conclusions drawn above probably have a very simple morphological basis. The depth of a nerve cell under the pia is the length of its main dendrite, being the dendrite that rises from the cell body and ends in the pia glia membrane. And in a cats brain the basal dendrites of a cortical neuron were found to be proportional to the volume of its nucleus. The quotient nucleus volume divided by the depth, thus, is proportional to the quotient of the length of the basal and the main dendrites, it is a value directly related to the shape of the dendrite field of the neuron: the relation between the height and the breadth of the dendrite field is a constant number of times that quotient. The conclusions drawn above, in consequence, may be formulated as follows: *in the upper group the shape of the dendrite fields varies normally, independently of and as many times as the length of the main dendrites* (with normal variation of the shape a normal distribution of the logarithm of the quotient breadth: height is meant).

The lower group shows two differences with the upper one.

In the diagram of *log* nucleus volume and *log* depth the regression line through the mean volumes has not a gradient with a tangent equal to 1 but nearly equal to 2: the mean nucleus volume increases proportional to a higher power of the depth. The second difference will be discussed later. The field takes the form of a circle when *log* (nucleus volume divided by the higher power of the depth) is plotted against the logarithm of the higher power of the depth. The radius of the circle is equal to that of the upper zone described above.

The neuron with a nucleus volume and a depth equal to the values of the centre of the circle may be called the central neuron of the field. The central neurons of both fields have exactly the same nucleus volumes: the centres were found to be lying exactly on the same vertical line (standard error of the measuring smaller than 5 %). The depth of the lower central cell is 3,16 $\times$ that of the upper one.

Since then measurements of other fields of the same specimen of cerebral cortex have been made. The various Nissl preparations were made simultaneously and ample precautions were taken that the various

parts cut from the cortex were under the same conditions during fixation, embedding, slide cutting and colouring.

The logarithms of the nucleus volumes and depths measured in 6 fields are shown in the figs. 3—8. All the relations described above can be found in these diagrams of the other architectonic fields, three dimensions only differ: the radius of the circle, the nucleus volume of the central neuron and the gradient of the regression line in the lower group. In all the architectonic fields this nucleus volume and the tangent of the gradient being proportional to that radius, these three varying values are dependent upon each other. The diagrams of these various architectonic fields, thus, differ in the value of one property only, indicated by the length the said radius.

In order to demonstrate this conclusion in an easy way the transformed circles (according to this conclusion) are drawn in the diagrams and a glance at the figures will demonstrate, that these lines describe in a fairly exact way all the fields of measuring points, notwithstanding the great differences between the architectonic fields studied.

The principle used in constructing these transformed circles is demonstrated in fig. 1, being a theoretical diagram of the logarithms of the

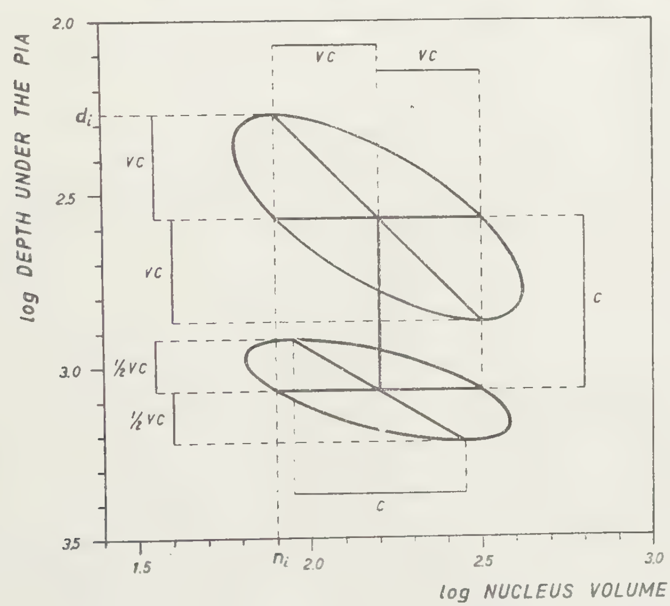


Fig. 1. General scheme of the size of the ganglion cells at the different depths of the cerebral neocortex. The value of the field exponent v varies from $\frac{1}{2}$ to 1 and characterizes each cyto-architectonic field. The values of c, d_i and n_i are nearly constant in the whole human neocortex.

nucleus volumes and depths in a random architectonic field of the human cerebral cortex.

The regression line of the upper group starts at a depth of 168 μ

(*log* 168 = 2,25) and a nucleus volume of 83 μ (*log* 83 = 1,92). (The table p. 953 and fig. 2 show that in all the architectonic fields the upper group begins — at the border between the first and second layer — with almost the same mean nucleus volume and at about the same distance from the pia. In the table the proportion of the measured values of that depth and volume and the constant values of the scheme are expressed by their logarithms p and q; these are small and they do not show a

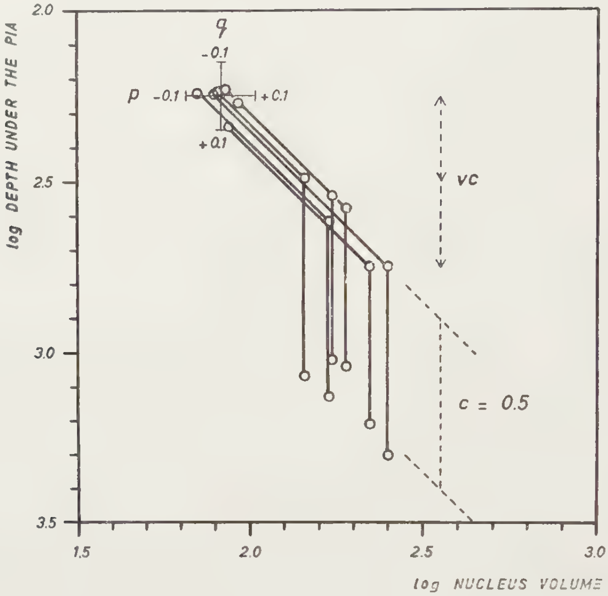


Fig. 2. The starting points of the upper regression line, the upper and the lower centre of the architectonic fields represented in figs. 3—8.

correlation to the field exponent, described below). The regression line has a gradient of exactly 45°. The centre of the upper field, therefore, lies at equal distances more to the right and lower down. This distance is the radius of the circle spoken of above and it varies in the various architectonic fields.

In the vertical line drawn through the upper centre the lower centre is found. In nearly all the architectonic fields studied the distance between these centres is found to be nearly the same (0,50). This means that in all the architectonic fields the central neuron of the lower group has the same nucleus volume as that of the upper group and a depth about 3,16 times larger than the upper one (*log* 3,16 = 0,50). In other vertebrates this factor has a different value. It thus being characteristic for the neocortex of a definite species of animal, it may be called the *cortex factor* (C). In man this cortex factor is 3,16.

The radius of the upper circle is found to be different in the various architectonic fields. It is maximal in the motoric area giganto-pyramidalis

	area striata	area parastriata	area orbitalis granulosa	area postcentralis granulosa	area paracentralis	area praecentralis gig. pyram.
exponent of the field (v)	0.50	0.54	0.70	0.76	0.82	1.00
log factor of the field ($vc = 0.5\, v$)	0.25	0.27	0.35	0.38	0.41	0.50
mean log nucleus volume at the upper border of upper group (n_i)	1.91	1.97	1.93	1.85	1.94	1.90
correction of volume ($p = n_i - 1.92$)	—0.01	+0.05	+0.01	—0.07	+0.02	—0.02
log nucleus volume central neurons (n_c) (equal to $1.92 + vc + p$)	2.16	2.24	2.28	2.23	2.35	2.40
log depth of upper border upper group (d_i)	2.24	2.27	2.23	2.24	2.34	2.25
correction of depth ($q = d_i - 2.25$)	—0.01	+0.02	—0.02	—0.01	+0.09	0.00
log depth of upper central neuron (d_I) (equal to $2.25 + vc + q$)	2.49	2.54	2.58	2.62	2.75	2.75
log depth of lower central neuron (d_{II})	3.07	3.02	3.04	3.13	3.21	3.30
$d_{II} - d_I$	0.58	0.48	0.44	0.51	0.46	0.55
tangent of upper regression line	1.00	1.00	1.00	1.00	1.00	1.00
tangent of lower regression line (equal to v)	0.50	0.54	0.70	0.80	0.82	1.00
number of neurons counted in upper group (under 0.01 mm^2 pial surface)	484	328	354	446	398	324
item in lower group	696	336	392	474	324	302
the proportion of these numbers	0.70	0.98	0.90	0.94	1.23	1.08

of the gyrus centralis anterior. In this area with the largest cells the radius measures 0,50, being equal to *log C* (the logarithm of the cortex factor). It is found minimal in the optic area striata, the field with the smallest cells. In this area it is 0,25 or $\frac{1}{2}\, c$ (if $c = \log C$). In the other areae studied its value lies between $\frac{1}{2}\, c$ and c. The radius, therefore, can best be expressed by the formula vc, in which c is the constant distance between the two centres, that is the logarithm of the cortex factor C. In the various areae the value of v varies from $\frac{1}{2}$ to 1.

What is the exact meaning of this v? Owing to the fact that it defines the size of the circles (the size of the correlation fields) it characterizes to a great extent the structure of each special architectonic field.

In the first place it defines the nucleus volume of the two central neurons. In the general scheme of fig. 1 the upper centre lies vc more to the right than the starting point of the upper regression line. In each area this starting point representing the same nucleus volume of 83 μ^3, the log

nucleus volume of the central neuron is $log\ 83 + vc = log\ 83 + v\ log\ C = log\ 83 + v\ log\ 3{,}16$. The nucleus volume of the central neurons, in consequence, is $3{,}16^{v} \times 83\ \mu^3$.

The central nucleus volume being nearly equal to the geometrical mean nucleus volume of the upper main group, the value of v of an architectonic field determines the mean size of its ganglion cells: a larger value of v is found in an area with larger cells. As in the formula $log\ N_c = 3{,}16^{v} \times 80\ \mu^3$ the v is an exponent, v can be called *the exponent of the architectonic field.*

In the second place v defines the depth of the two central nerve cells. In fig. 1 the centre of the upper field lies vc lower than the starting point of the regression line. The starting point lying at a depth of 168 μ, the depth of the upper central nerve cell is $3{,}16^{v} \times 168\ \mu$. In an area with larger nerve cells the upper central neuron has a larger main dendrite than in an area with smaller cells.

In each area the depth of the lower central neuron being 3,16 times the depth of the upper one, the depth of the lower central cell is $3{,}16 \times 3{,}16^{v} \times 168\ \mu$ or $3{,}16^{v+1} \times 168\ \mu$.

In the third place it defines the thickness of the two main groups.

From fig. 1 it follows, that the lower border of the upper group is indicated $2vc$ lower than its upper border (the starting point). The lower border, thus, lies at a depth equal to $3{,}16^{2v} \times 168\ \mu$.

The height of the lower field in fig. 1 is half as large as that of the upper field. Its upper border is drawn $\frac{1}{2}vc$ higher and its lower border $\frac{1}{2}vc$ lower than its centre. The lower main group, thus, starts at a depth of $3{,}16^{\frac{1}{2}v-1} \times 168\ \mu$ and it ends at a depth of $3{,}16^{1\frac{1}{2}v+1} \times 168\ \mu$. The depth of the lower border of the lower group is the same as the total thickness of the cortex. The field exponent v, thus, also defines the thickness of the cortex in the field.

In the fourth place v defines the geometrical mean volume at each depth of the lower group.

In the upper group the mean volume is determined by the depth only: the starting point of the regression line indicates the constant mean nucleus volume of 83 μ^3 at a depth of 168 μ and the mean nucleus volume in the upper group being proportional to the depth, in each area the mean nucleus volume at a depth of $d\ \mu$ in the upper group is $\frac{d}{168} \times 83\ \mu^3$

In the lower group, however, the nucleus volume increases proportional to a higher power of the depth, the regression line having a smaller gradient than 45°. This gradient differs in the different architectonic fields. Its tangent is found equal to v. In the lower group, thus, the depth increases proportional to the v-power of the mean nucleus volume. The value of v being known, the mean nucleus volume at each depth of the lower group can be calculated from it.

Owing to the height of the lower correlation field in fig. 1 being vc and the tangent of its regression line being v, the endpoint of this regression line lies c more to the right than its starting point: in each area the largest mean nucleus volume of the lower group (lying at the border between cortex and white matter) is 3,16 times as large as its smallest mean nucleus volume (lying at the upper border of the lower group). The total variation range of the mean nucleus volumes in the lower group, expressed in times, thus, is the same in all areae. In the upper group, on the contrary, it differs in the various areae, the endpoint of the regression line lying $2vc$ more to the right than its starting point, the largest mean nucleus volume, in other words, being $3{,}16^{2v}$ times as large as the smallest one.

In the fifth place the field exponent v defines the range of the variation in size of the ganglion cells at each depth in the cortex, the maximal variation at a special depth being determined by the horizontal diameter at that depth of the correlation field, bordered by the transformed circles of fig. 1. (More exactly it follows from the standard deviation spoken of above, which is equal to 0,37 times the radius or equal to 0,185 v.)

In the sixth place the exponent seems to define the number of ganglion cells present in each group under a unit of pial surface. In the upper group this number seems to be fairly constant, in the lower group it seems to vary inversely proportional to v. The numbers of cells measured, however, are to small to make this certain.

The details of the general scheme of fig. 1 described above can be summarized as follows.

In each architectonic area of the human cerebral cortex an upper and a lower main group of nerve cells can be distinguished. The cells of each group can be seen as variations from a mean type, called the central neuron of the group. The upper and the lower central neuron of one area have the same nucleus volume. The main dendrite (= depth under the pia) of the lower one is 3,16 times as long as in the upper one.

In the upper group the quotient nucleus volume divided by depth ($N:D$, probably being a measure of the shape of the dendrite field, in a cat N being found proportional to the length of the basal dendrites and D being the length of the main dendrite) varies normally (for exact meaning see above) and as many times as and independently of the depths (length of the main dendrites). The maximal extension of these variations from the values of the central neuron is $3{,}16^{v}$ times, in which v is the exponent of the field. In the various fields the value of v varies between $\frac{1}{2}$ and 1.

In the lower group the quotient $N:D^{1/v}$ varies normally and independently of the depth and as many times as $N:D$ in the upper group.

In consequence of these types of variation the mean nucleus volume varies with the depth: in the upper group it increases proportional to the depth, in the lower group proportional to the $1/v$ power of the depth.

Another consequention of these types of variation is the possibility to express the thickness of the cortex and of the two main groups and probably the number of the neurons as a function of the field exponent v.

The field exponent v, thus, defines the various sizes of the neurons at each depth of the cortex, it thereby describes the structure of the cyto-architectonic field to a great extent.

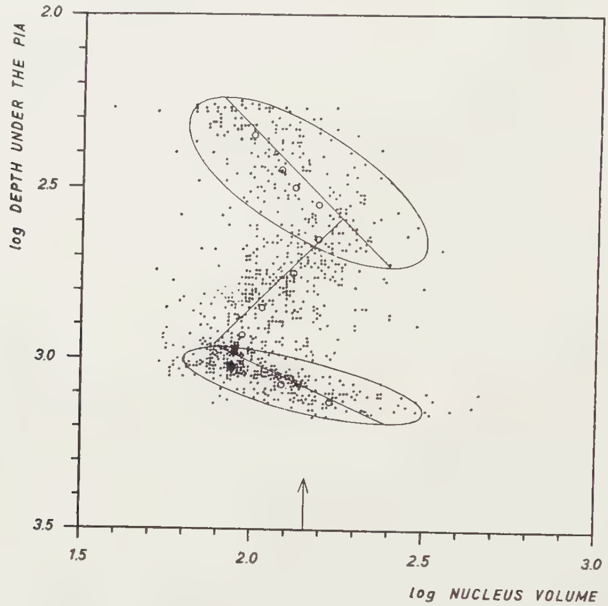


Fig. 3. Area striata ($v = 0.50$).

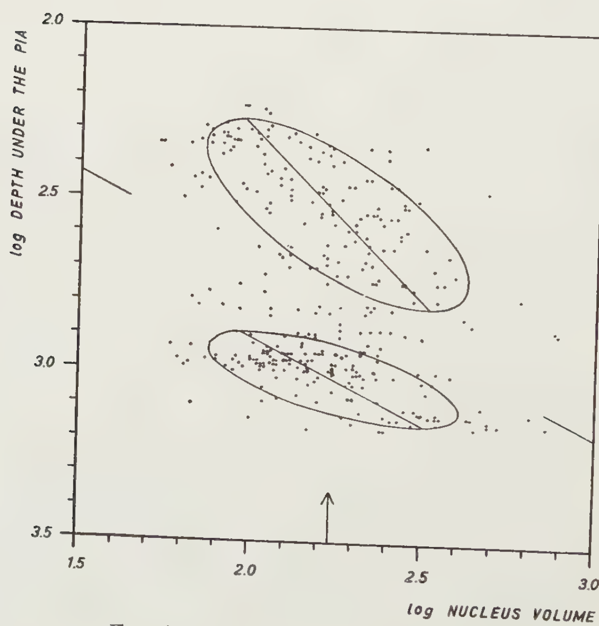


Fig. 4. Area parastriata ($v = 0.54$).

Indeed the cyto-architectonic fields, thus, seem to be different manifestations of one common scheme of structure and by far the most of their differences are due to the variation in size of only one property of that scheme. This size can be expressed by the value of the so called field exponent (v). Differences of smaller size, present in the measurings, could be defined by two other exponents p and q. The size of these extra

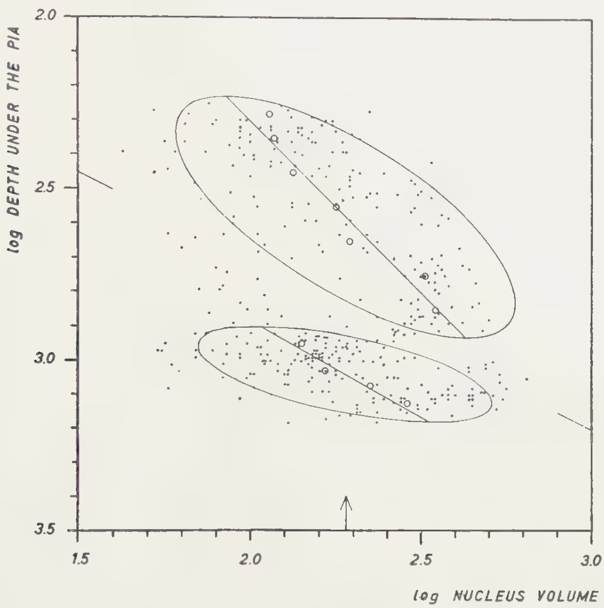


Fig. 5. Area orbitalis granulosa ($v = 0.70$).

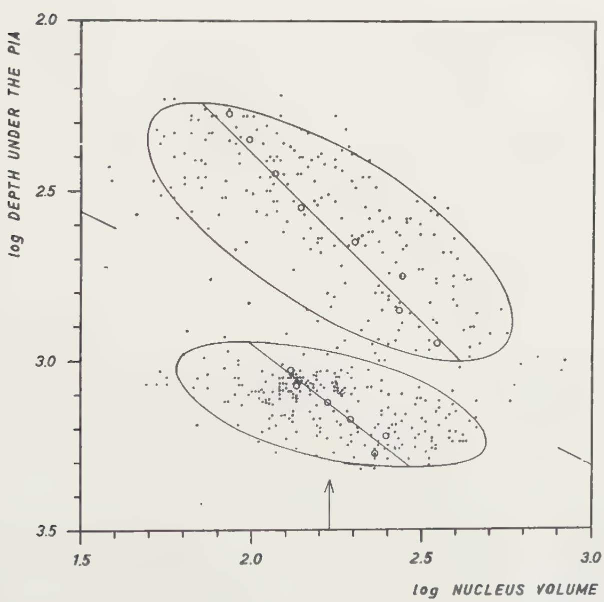


Fig. 6. Area postcentralis granulosa ($v = 0.76$).

exponents, however, is so small that it is not certain if they express real architectonic differences between the fields. If so, the differences indi-

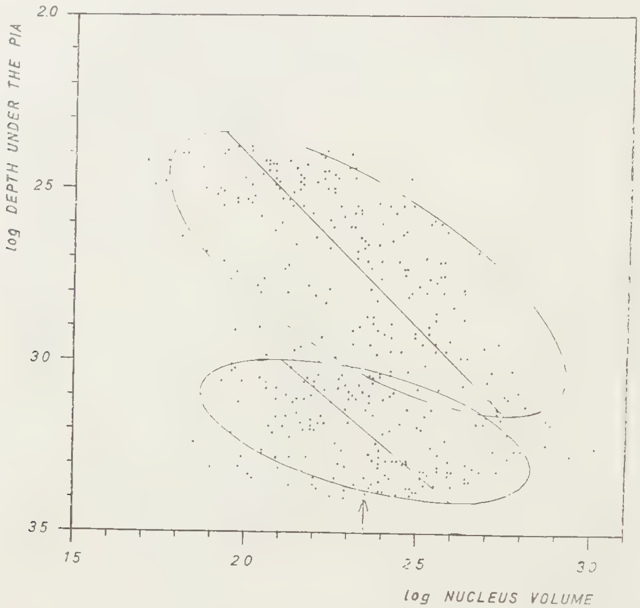


Fig. 7. Area paracentralis ($v = 0.82$).

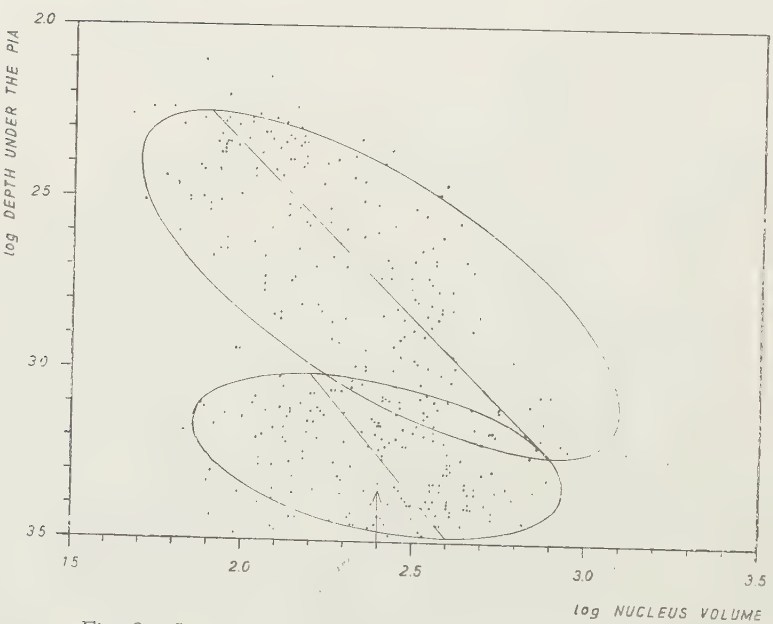


Fig. 8. Area praecentralis giganto-pyramidalis ($v = 1.00$).

cated by p and q are far smaller than those indicated by the field exponent v.

Anthropologie. — *Een onderkaaksfragment van Elephas primigenius met menschelijke bewerking.* Door A. J. P. v. d. BROEK. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of November 25, 1939.)

Uit de zandgraverij te Maarn is een onderkaaksfragment te voorschijn gekomen, dat onze aandacht waard is.

De spoorweginsnijding bij Maarn gaat door den glacialen stuwwal, welks kern uit gestuwd praeglaciaal (prae-Riss) materiaal bestaat. Hij wordt hier en daar door resten van het keizand der Riss-periode bedekt.

Voor zoover uit de mededeeling van de werklieden en uit aanwijzing van de vindplaats is af te leiden, is het onderkaaksfragment afkomstig uit de geplooide lagen van den stuwwal, zoodat deze, naar den tijd, in het praeglaciaal moet worden gesteld, d.w.z. den tijd voorafgaande aan de grootste uitbreiding van de ijsbedekking van den Riss-ijstijd.

Het fragment is een deel van het corpus mandibulae. Aan de voorzijde gaat de breuk vlak langs het foramen mentale internum, aan het achtereinde is een klein gedeelte van den ramus ascendens aanwezig.

In verband met den geologischen ouderdom van de laag, waaruit deze kaak te voorschijn is gekomen, moet de vraag beantwoord worden of wij met de kaak van E. antiquus, dan wel van E. primigenius te doen hebben.

Aan de buitenzijde (fig. 1) komen 3 foramina mentalia voor, aan de binnenzijde (fig. 2) één. De nog aanwezige rest van den ramus ascendens maakt een stompen hoek met den bovenrand van het corpus mandibulae, wat er voor zou pleiten, dat wij met een betrekkelijk jong individu te maken hebben; hoewel in de richting van corpus en ramus ascendens ten opzichte van elkaar, vooral bij E. primigenius, variabele verhoudingen bestaan (POHLIG).

Aan de binnenzijde (fig. 2) zijn de afdrukken van hoogstwaarschijnlijk twee gebitselementen te zien, die wel bewijzen, dat het fragment van E. primigenius afkomstig is.

Vooraan vindt men een diepe, gekromde alveolus, waarvan de doorsnede aan het boveneinde 35 mm. is, de diepte, in rechte lijn gemeten, 95 mm. Daarachter vindt men een trapeziumvormigen indruk, aan den bovenrand ± 145 mm. lang, in 't midden 95 mm. hoog. Duidelijk zijn hier de indrukken van 10 (11?) lamellen vast te stellen, de formule van het desbetreffende element zou dus moeten luiden (x) 10 (11?) (x); met een kroonlengte (gemeten volgens de opgave van POHLIG) van hoogstens 135 mm. De afstand der lamellen is ± 12 mm.

In het onderstaande schema zijn de formules en de maten van enkele

elementen uit het gebit van E. antiquus en E. primigenius volgens POHLIG's opgaven samengevoegd.

	E. primigenius	E. antiquus
m_2	x 6 x — x 8 x 52 — 71 mm.	x 6 x — x 8 x 57 — 75 mm.
m_3	x 9 x — x 12 x 85 — 116 mm.	x 7 x — x 8 x 85 — 110 mm.
M_1	x 11 x — x 15 x 113 — 150 mm.	x 9 x — x 12 x 155 — 172 mm.

Uit de bovenstaande maatverhoudingen volgt wel, dat wij niet met E. antiquus te maken kunnen hebben. Op grond van de formule zou het element slechts M_1 kunnen zijn, en hiermede komt de maat niet overeen. De lamellen zijn bij E. antiquus veel breeder dan bij E. primigenius.

Wat E. primigenius betreft kan men aan drie mogelijkheden denken, n.l. m_2 en m_3; m_3 en M_1; of alveolus en indruk als twee gedeelten van één element. Dit zou dan m_2 of m_3 moeten zijn. Hiertegen pleit echter de kroonlengte, die voor m_3 ten hoogste 116 mm. bedraagt, terwijl bij dit object de lengte ± 135 mm. bedraagt. Mijns inziens komt dus slechts in aanmerking m_3 (rest) en M_1; onwaarschijnlijker acht ik m_2 (rest) en m_3. Dit is niet van veel belang, waar POHLIG mededeelt, dat er geen scherpe onderscheidingskenmerken zijn tusschen m_3 en M_1 (l.c. blz. 124).

De beteekenis dezer vondst ligt niet in den ouderdom van het dier, als hierin, dat 1e Elephas primigenius in Nederland in hoofdzaak in lagen beantwoordend aan den Würm-ijstijd is gevonden; dit dus één van de zeldzamer vondsten uit den prae-Risstijd is en 2e, dat het kaakfragment duidelijk sporen draagt, die slechts aan menschelijke bewerking kunnen worden toegeschreven.

Deze sporen bestaan in een aantal diepe en ondiepe krassen en inkepingen in de omgeving der plaats, waar het corpus mandibulae in den ramus ascendens overgaat. Plaats zoowel als richting dezer krassen zijn zóó, dat men zich niet aan den indruk onttrekken kan, dat men te doen heeft met eene bewerking, die ten doel had den M. masseter te verwijderen. De diepe inkepingen convergeeren naar de hoekplaats tusschen corpus en ramus, een aantal ondiepere langere krassen zijn op de buitenoppervlakte van het corpus mandibulae aanwezig. Zij zijn beperkt tot het achterste gedeelte van het fragment, op het voorste gedeelte ontbreken zij ten eenenmale. Aan de bovenzijde van het fragment is een stuk van de kaak met een scherpen rand afgebroken (fig. 2). Het maakt den indruk alsof dit met kracht van de kaak is afgeslagen.

Vier mogelijkheden voor het ontstaan dezer inkepingen en krassen moeten onder het oog worden gezien, n.l. 1e de krassen enz. moeten aan natuurlijke oorzaken worden toegeschreven, 2e zij zijn recent, 3e zij zijn door een (roof)dier gemaakt en 4e zij zijn door den mensch, tijdgenoot van dezen E. primigenius, aangebracht.

De eerste mogelijkheid kan worden uitgesloten op grond van plaatsing en richting der krassen en op de overweging, dat zij op de andere gedeelten van de buitenoppervlakte der kaak ontbreken.

De tweede mogelijkheid kan ook worden uitgesloten. In de eerste plaats ontving ik de zeer positieve verzekering van den vinder, dat de kaak met het zand naar beneden gekomen was en overhandigd in den toestand, waarin zij uit het zand is gekomen. Met name was de kaak niet met een steekschop aangeraakt of bewerkt. Enkele inkepingen, door mij, met een scherp voorwerp (beitel) gemaakt, geleken eenigermate op de groeven, waarvan boven sprake was, doch verschilden er in ander opzicht toch weer zoodanig van, dat aan de oudheid der bewerking niet viel te twijfelen.

Ook de mogelijkheid, dat de inkervingen door een roofdier gemaakt zouden zijn, kan men uitsluiten. Dat een groot roofdier dit stuk kaak slechts zoo oppervlakkig zou hebben geraakt en niet meer zou hebben vernietigd, is bijna uit te sluiten. De breukvlakken trouwens wekken geen enkele gedachte daaraan, noch geven zij indrukken van tanden te zien. Trouwens zou de indruk van een hoektand of scheurkies geheel anders zijn, dan deze oppervlakkige inkervingen en groefjes. Eindelijk zou men onder- en bovenkaaksindrukken van het roofdier moeten waarnemen.

Er blijft dus geen andere dan de laatste mogelijkheid over, dat de inkervingen en groeven producten van menschelijke behandeling zijn en wel van een mensch, die gelijktijdig met dezen mammoet leefde.

Samenvattend hebben wij dus te doen met een fragment van de rechter onderkaakshelft van Elephas primigenius, afkomstig uit een, aan den prae-Risstijd beantwoordende, zandlaag en bewerkt door den mensch.

Aanwijzingen omtrent de aanwezigheid van den mensch in Nederland tijdens den Riss-ijstijd zou men kunnen ontleenen aan eene mededeeling van v. d. Vlerk en Florschütz (1938), die eene kleiachtige zandlaag in Wezep, waaruit silex artefacten te voorschijn zijn gekomen, als beantwoordend aan het tweede maximum van den Riss-ijstijd beschouwen.

Bursch (1938) heeft deze stukken vergeleken met vondsten uit Oldebroek en uit Vollenhove. Volgens hem moet men in deze steenen de opeenvolging van drie kultuurphasen zien, n.l. Oldebroek (Acheuléen-Clactonien), Wezep (Clactonien), Vollenhove (Levalloisien). Hij meent, dat de vondsten in Wezep ouder moeten zijn, dan v. d. Vlerk en Florschütz opgeven. Nadere onderzoekingen moeten wij nog afwachten.

Andere vondsten van artefacten uit het z.g. oud-palaeolithicum (Bathmen, St. Geertruide) zijn niet genoegzaam geologisch vastgelegd, dat wij ze kunnen gebruiken.

Jong-palaeolithische vondsten zijn welbekend, zij beantwoorden aan den Würm-ijstijd en het post-glaciaal.

De vraag is gewettigd of de mensch, die dus klaarblijkelijk reeds vóór

het maximum van den Riss-ijstijd in Nederland aanwezig geweest is, tot het ras van Neanderdal heeft behoord.

Volgens moderne opvatting zou de mensch in Europa reeds in den Günz-ijstijd, resp. in het Günz-Mindel-interglaciaal, in Europa aanwezig geweest zijn. Kultuurphasen, die men algemeen als karakteristiek voor den Cro-Magnon mensch beschouwt, treden eerst in den Würm-ijstijd op; zoodat het grootste deel van het diluvium aan den Neanderdaler moet hebben behoord. In dit zeer lange tijdsgewricht valt ook de boven beschreven vondst.

Het onderkaaksfragment wordt bewaard in het Anatomisch Laboratorium der Rijks-Universiteit te Utrecht.

SUMMARY.

A description is given of a part of the mandible of Elephas primigenius, found in the preglacial sands at Maarn. As Elephas primigenius has seldom been found in the Netherlands in preglacial layers, a minute study of the object was necessary to ascertain that we had not to do with Elephas antiquus. Herefore were used the impressions of the molars at the inner side of the mandible.

The value of this object is that it bears scratches and notches, especially at the place of insertion of the M. masseter, which must have been made by man. An investigation of these scratches proved that they were not recent, not made by natural causes, nor even made by an animal.

This object proves that man (Neanderthal?) must have been present in the Netherlands at a time before the Riss-glaciation.

LITERATUUR.

BURSCH, F. C., FLORSCHÜTZ, F. and V. D. VLERK, I. M., An early palaeolithic site on the northern Veluwe. Proceedings Kon. Ned. Akad. v. Wetenschappen, Bd. **41**, 909 (1938).

BURSCH, F. C., Die vorneolithischen Kulturen in den Niederlanden, Geologie en Mijnbouw, I (1939).

KRUIZINGA, P., De insnijding bij Maarn. Jaarboek der Vereen. tot behoud van Natuurmonumenten (1918—1922).

POHLIG, H., Dentition und Kraniologie des Elephas antiquus Falc. mit Beiträgen über Elephas primigenius Blum. und Elephas meridionalis Nesti. Verhandlungen der Leopoldinisch-Carolinischen deutschen Akad. d. Naturforscher, Bd. **53**, **57** (Halle 1889, 1892).

RUTTEN, C. M. R., Die diluvialen Säugetiere der Niederlande, Proefschrift, Utrecht (1907).

A. J. P. v. d. BROEK: Een onderkaaksfragment van Elephas primigenius met menschelijke bewerking.

Fig. 1

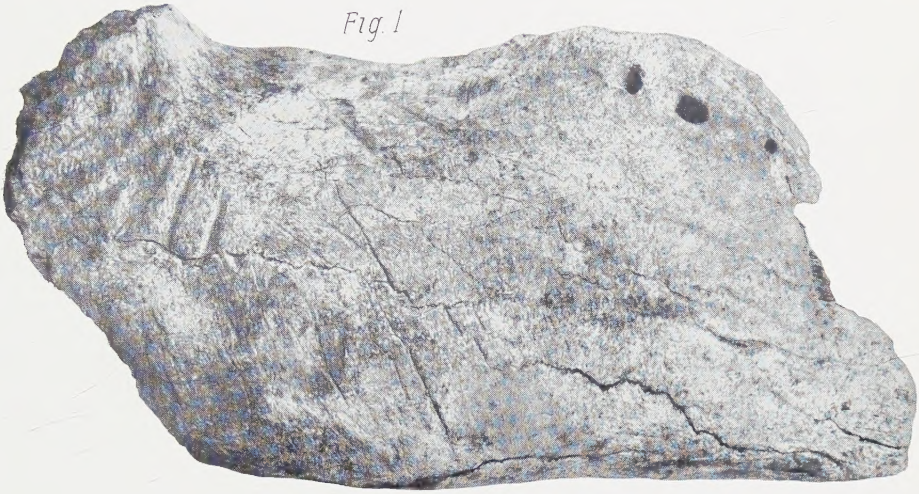


Fig. 2